①

Erdős-Sierpiński Problem:

Find solutions to $\sigma(n) = \sigma(n+1)$ (where $\sigma(n) = \sum_{d|n} d$.)

E.g.

$\sigma(14) = \sigma(15) = 24$ (*)

$\sigma(206) = \sigma(207) = 312$ (*)

$\sigma(957) = \sigma(958) = 1440$

1334

1364

1634

2685

⋮

18,873 (**)

⋮

19,358 (*)

⋮

174,717 (**)

⋮

5,559,060,136,088,313 (**)

⋮

Guy-Shanks

(*)

$n = 2p \qquad n+1 = 3^m q$

w/ $q = 3^{m+1} - 4$, $p = \dfrac{3^m q - 1}{2}$ prime

(**)

$n = 3^m q \qquad n+1 = 2p$

w/ $q = 3^{m+1} + 10$, $p = \dfrac{3^m q + 1}{2}$ prime

(2)

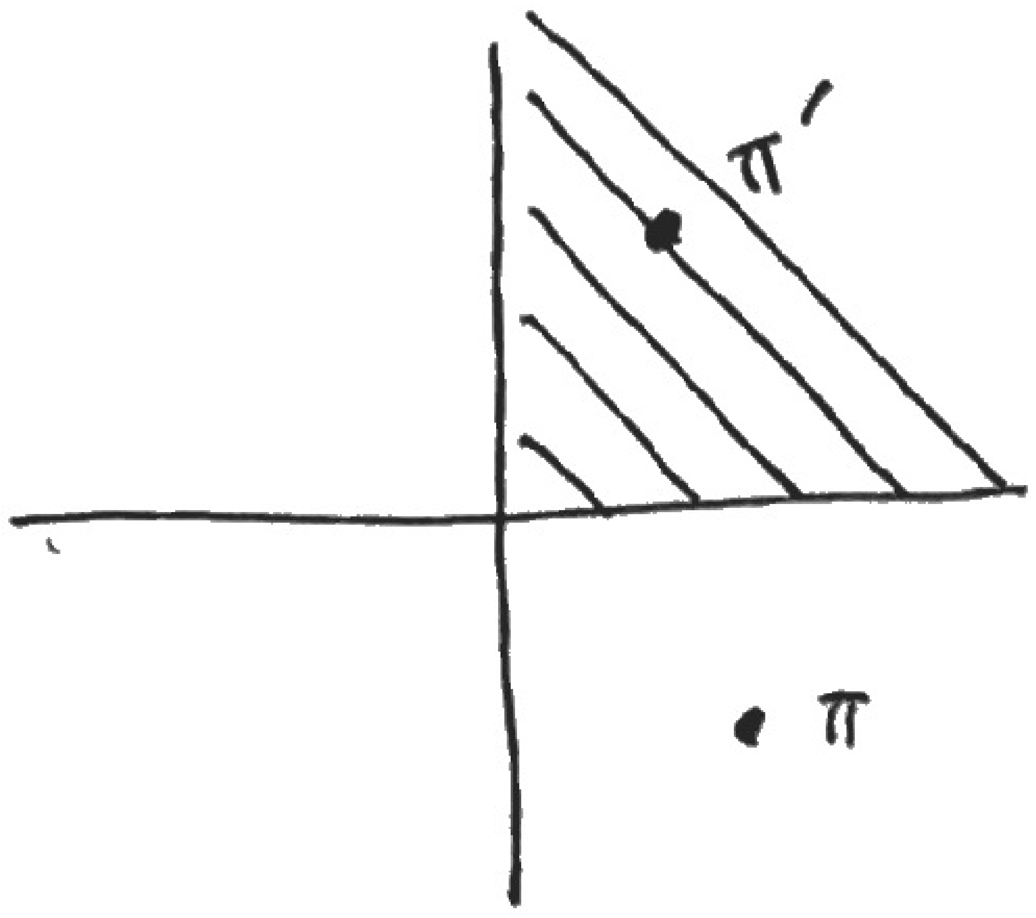
$\sigma(2p) = \sigma(3^n q)$ in (*) and (**).

note $\sigma(2) = 3$, $\sigma(3) = 2^2$

seamless segue

R. Spira (1961) defined σ on $\mathbb{Z}[i]$:

If π is prime, let π' be an associate of π in the 1st Quadrant, not on the imaginary axis.



Then $\sigma(\pi^k) = \dfrac{(\pi')^{k+1} - 1}{\pi' - 1}$

Then if $\eta = \varepsilon \pi_1^{k_1} \pi_2^{k_2} \cdots \pi_n^{k_n}$

with each π_i in Q1 ∖ Im and ε some unit:

$$\sigma(\eta) = \prod \frac{\pi_i^{k_i+1} - 1}{\pi_i - 1}.$$ So it's multiplicative.

Note This doesn't agree with usual σ on $\mathbb{Z}$:

normal: $\sigma(5) = 6$

abnormal: $\sigma(5) = \sigma(2+i)\,\sigma(2-i) = \sigma(2+i)\,\sigma(1+2i)$

$\underbrace{(2+i)}_{Q1}$ $\underbrace{(2-i)}_{Q4,\ = -i(1+2i)}$

$= ((2+i)+1)((1+2i)+1) = 4 + 8i \neq 6$

Spira sought "Gaussian perfect" numbers $\sigma(n) = (1+i)n$

smallest prime.

③ Two smallest primes: $1+i$, $2+i$

$\sigma(1+i) = 2+i$ $\quad$ $\cancel{\sigma(2+i) = 3+i}$ $\quad$ $\sigma(1+2i) = 2+2i = \varepsilon(1+i)^3$

conjugate

So $1+i$, $2+i$ share the relationship in $\mathbb{Z}[i]$ enjoyed by $2+3$ in $\mathbb{Z}$.

Sooo....

The E.-S. problem in $\mathbb{Z}[i]$:

Find solutions to $\sigma(n) = \sigma(n+1)$

$$\sigma(n) = \sigma(n+i)$$

and, heck, why not $\sigma(n) = \sigma(n+(1+i))$

Eg's

$$\sigma(3+7i) = 10+10i = \sigma(4+7i)$$

$$\sigma(19+25i) = -20+60i = \sigma(20+25i)$$

and

$$\sigma(15+9i) = 40i = \sigma(15+10i)$$

$$\sigma(25+39i) = -40+80i = \sigma(25+40i)$$

and $\sigma(5+5i) = 20i = \sigma(6+6i)$

$$\sigma(13+4i) = 20i = \sigma(14+5i)$$

(4)

Following Guy–Shanks, change 2, 3 to $1+i$, $2+i$. Find solutions of the form $\sigma(\frac{(1+i)^p}{\gamma_p}) = \sigma(\frac{(2+i)^q}{\gamma_q})$, p, q Gaussian primes

$$(1+\tfrac{i}{2})p = n \quad (2+\tfrac{i}{2})q = n \pm 1, \pm i, \pm(1+i).$$

ONLY 0 or 1 solution in each case.

<u>Theorem</u> Fix Gaussian primes p, q. Then there is at most one solution to $\sigma(pr) = \sigma(qs)$ with r, s prime and $qs = pr + \varepsilon$, $\varepsilon = \pm 1, \pm i, \pm(1+i)$

<u>proof</u> Let $p = a+bi$, $q = c+di$, $r = e+fi$, $s = g+hi$

Plug into the equations $\sigma(pr) = \sigma(qs)$

and $qs = pr + \varepsilon$

Equate real + imaginary parts $\rightarrow$ 4 equations in e, f, g, h.

Either a unique solution or no solution.

Continuing work: Solve $\sigma(n) = \sigma(n+k+i)$

Similar work in $\mathbb{Z}[\omega]$ (Eisenstein Integers) and other complex Quadratic extentions.

$$(1+i)(a+bi) = (2+i)(a+1+bi)$$
$$(a-b) + i(a+b) = (2a+2-b) + i(a+1+2b)$$
$$a = -2, b = -1$$

Once again forces a contradiction since q is outside of the first quadrant.

As shown there are no solutions of these four cases. How about the cases where both z and $z+1$ are the product of two distinct primes?

4. Solutions to $g\sigma(pq) = g\sigma(rs)$

Guy and Shanks used the form:

$$\sigma(2) = 3, \quad and \quad \sigma(3) = 4 = 2^2$$ \mbox{ where }

Likewise we will extend this to the Guassian Integers:

$$g\sigma(1+i) = 2+i \quad where \quad N(2+i) = 5 = N(1+2i) \quad and \quad g\sigma(1+2i) = 2+2i = (1+i)^3$$

Furthermore, following Guy and Shanks method with Gaussian Integers we have z and $z+1$ the form of two distinct primes. Let $z = pq = (a+bi)(c+di)$ and $z+1 = rs = (e+fi)(g+hi)$. We seek solutions for:

$$g\sigma(z) = g\sigma(z+1)$$
$$g\sigma((a+bi)(c+di)) = g\sigma((e+fi)(g+hi))$$
$$g\sigma(a+bi)g\sigma(c+di) = g\sigma(e+fi)g\sigma(g+hi)$$
$$(a+1+bi)(c+1+di) = (e+1+fi)(g+1+hi)$$
$$(ac+a+c+1-bd)+i(bc+b+1+ad+d) = (eg+e+g+1-fh)+i(fg+f+eh+h)$$

Giving us two equations:

$$ac+a+c-bd = eg+e+g-fh$$
$$bc+b+ad+d = fh+f+eh+h$$

Also recall

$$z+1 = (a+bi)(c+di)+1 = (e+fi)(g+hi)$$
$$(ac-bd+1)+i(ad+bc) = (eg-fh)+i(eh+fg)$$

Giving us two more equations:

$$ac-bd+1 = eg-fh$$
$$ad+bc = eh+fg$$

Suppose we know a, b, e, f, then we can solve for c, d, g, h by our four equations:

$$ac+a+c-bd = eg+e+g-fh$$
$$ad+d+bc+b = eh+fg+h+f$$
$$ac-bd = eg-fh-1$$

$$bc + ad = fg + eh$$

Thus our solutions:

$$c = \frac{ea^2 - 2e^2a - ea - a + eb^2 - bf - 2ebf + ef^2 + f^2 + e^3 + e^2 + e}{a - e^2 + b - f^2}$$

$$d = \frac{a^2b - 2eab - af + b^3 - 2b^2f + bf^2 + e^2b + eb + b - f}{a - e^2 + b - f^2} - b + f$$

$$g = \frac{a^3 - 2ea^2 - a^2 + ab^2 - 2abf + af^2 + e^2a + ea - a - b^2 + bf + e}{a - e^2 + b - f^2}$$

$$h = \frac{a^2b - 2eab - af + b^3 - 2b^2f + e^2b + eb + b - f}{a - e^2 + b - f^2}$$

Examples of where $g\sigma(z) = g\sigma(z+1)$

$$g\sigma(3+7i) = 10 + 10i = g\sigma(4+7i)$$
$$g\sigma(19+25i) = -20 + 60i = g\sigma(20+25i)$$
$$g\sigma(19+75i) = -100 + 100i = g\sigma(20+75i)$$
$$g\sigma(40+85i) = -100 + 140i = g\sigma(41+85i)$$
$$g\sigma(90+245i) = -320 + 480i = g\sigma(91+245i))$$

5. Non-Consecutive Gaussian Integers Containg Two Primes

For $g\sigma(z) = g\sigma(z + k + qi)$ can also be solved using the previous formula. It is easy to show using the previous equations.

$$n = (a+bi)(c+di), \quad z + k + qi = (e+fi)(g+hi)$$

and by $g\sigma$ of primes

$$g\sigma(z) = (a + bi + 1)(c + di + 1) = (ac + a + c - bd) + i(ad + d + bc + b)$$

$$g\sigma(z + k + qi) = (e + fi + 1)(g + hi + 1) = (eg + e + g - fh) + i(eh + fg + h + f)$$

and we have the four equations

$$ac - bd = eg - fh - k$$
$$bc + ad = fg + eh - q$$
$$ac + a + c - bd = eg + e + g - fh$$
$$ad + d + bc + b = eh + fg + h + f$$

Giving the solutions: $c = \frac{-ea^2 - afq + eak + ak + 2e^2a - eb^2 + bfk + 2ebf + bq + ebq - f^2k - ef^2 - fq - e^2k - ek - e^3}{a^2 - 2ea + b^2 - 2bf + f^2 + e^2}$