On a divisibility relation for Lucas sequences

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The objects of the investigation: Lucas Sequences

- Lucas sequence and its companion:
  $\mathbf{U} := \mathbf{U}(a,b) = \{U_n\}_{n\geq 0}$, $U_0 = 0,\ U_1 = 1$ and

$$U_{n+2} = aU_{n+1} + bU_n \quad \text{for all} \quad n \geq 0, \quad b \in \{\pm 1\}. \qquad (1)$$

- We put $\mathbf{V}(a,b) = \{V_n\}_{n\geq 0}$ for the Lucas companion of $\mathbf{U}$:
  $V_0 = 2,\ V_1 = a$, same recurrence;
- Characteristic equation is $x^2 - ax - b = 0$ with roots

$$(\alpha,\beta) = \left(\frac{a+\sqrt{a^2+4b}}{2}, \frac{a-\sqrt{a^2+4b}}{2}\right).$$

- The Binet formulas for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n \qquad \text{for all} \qquad n \geq 0.$$

- Assume $\Delta = a^2 + 4b > 0$ and that α/β is not a root of unity (that is, $(a,b) \notin \{(0,\pm 1),\ (\pm 1,-1),\ (2,-1)\}$).

Diophantine equations involving binary sequences I

- If $a = b = 1$, we get the Fibonacci, resp., Lucas sequences.
- In this case, Cohn and, independently, Wyler (both in 1964) proved that $U_n = \square$ iff $n = 0, 1, 2, 12$. Cohn slightly generalized this result.
- McDaniel and Ribenboim (1992) showed (using divisibility methods) that if $U_n = \square, 2\square$, then $n \leq 12$.
- Mignotte and Pethö (1993) using linear forms in logarithms showed that if $b = -1, n > 4$, then $U_n = w\square$ is impossible if $w \in \{1, 2, 3, 6\}$, and these equations have solutions for $n = 4$ only if $a = 338$, and then, $U_4 = (2 \cdot 13 \cdot 239)^2$.

Diophantine equations involving binary sequences II

- Nakamula and Pethö (1998) used the same method to investigate when $U_n = w\Box$ for $b = 1$, $w \in \{1, 2, 3, 6\}$. They showed that $n \le 2$, except when $(a, n, w) = (1, 12, 1), (1, 3, 2), (1, 4, 3), (1, 6, 2), (2, 4, 3), 2, 7, 1), (4, 4, 2)$.
- Regarding the companion **V**, they showed that if $V_n = w\Box$, when $w \in \{1, 2, 3, 6\}$, then $n \le 1$, when $b = 1$ and a is even; and when $b = -1$, then $n \le 1$, except when $(a, n, w) = (1, 2, 3), (1, 3, 1), (1, 6, 2), (2, 2, 6), (3, 3, 36)$.

Diophantine equations involving binary sequences III

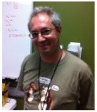
- Stewart (1982) found an effective finiteness result for shifted perfect powers in binary recurrence sequences: if the equation $U_k = x^n + c$ has a solution in integers x, n, c and k, with $n \geq 2$ and $|x| > 1$, then, under some conditions, $\max\{|x|, n\}$ is bounded above effectively in terms of c and the recurrence.
- Recall the celebrated result of Y. Bugeaud, M. Mignotte, S. Siksek (2006): The only powers in $\{F_n\}_n$ are $F_0 = 0, F_1 = F_2 = 1, F_6 = 8, F_{12} = 144$;

Diophantine equations involving binary sequences IV

- Making Stewart's result more precise, Bugeaud, Luca, Mignotte, Siksek (2008) showed that $F_n \pm 1 = y^p$, $p \geq 2$, then $(n, \pm 1, y, p) = (0, 1, 1, p)$, $(4, 1, 2, 2)$, $(6, 1, 3, 2)$, $(1, -1, 0, p)$, $(3, -1, 1, p)$, $(5, -1, 2, 2)$.
- More recently, Bennett, Dahmen, Mignotte, Siksek (2014) proved a diophantine eq. result that can potentially be applied to the equations $U_k = w\,x^n + c$ and $V_k = w\,x^n + c$, as well as, $F_k \pm F_{2j} = ax^n$.

Diophantine equations involving binary sequences V

- In 2012/2013 Komatsu, Luca, Tachiya:

Assume that m and n are coprime, so, F_n and F_m are coprime, thus F_{n+1}/F_n is defined modulo F_m. They showed that the congruence class $F_{n+1}/F_n \pmod{F_m}$ has multiplicative order s modulo F_m and $s \notin \{1, 2, 4\}$, then

$$m < 500s^2. \tag{2}$$

Comments the KLT result I

- It is possible that this order is $s = 1$. It happens precisely when $F_{n+1} \equiv F_n \pmod{F_m}$, so $F_m | F_{n+1} - F_n = F_{n-1}$, which holds when $m | n - 1$, that is, $n \equiv 1 \pmod{m}$.
- It is also possible that $s = 2$, since in this case, $F_{n+1}^2 \equiv F_n^2 \pmod{F_m}$, so $F_m | F_{n+1}^2 - F_n^2 = (F_{n+1} - F_n)(F_{n+1} + F_n) = F_{n-1} F_{n+2}$.
- Let $m > 12$. By Primitive Divisor Theorem (Bilu-Hanrot-Voutier, 2001) (actually, Carmichael's Theorem from ("*On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$*", Ann. Math. (2) 15 (1913), 30–70) would be sufficient, F_m has a primitive prime factor p, that is, $p \mid F_m$, but $p \nmid F_\ell$, for $1 \le \ell < m$.
- Thus, either $p \mid F_{n-1}$ or $p \mid F_{n+2}$. When $m \mid n - 1$, we recover the $s = 1$ case, and so, it must be that $n \equiv -2 \pmod{m}$.

Comments on the KLT result II

- It is also possible to have $s = 4$. In this case $F_{n+1}^4 \equiv F_n^4 \pmod{F_m}$. Thus

$$F_m \mid F_{n+2}^4 - F_n^4 = (F_{n+1} - F_n)(F_{n+1} + F_n)(F_{n+1}^2 + F_n^2)$$
$$= F_{n-1} F_{n+2} F_{2n+1}.$$

- Again, if $m > 12$, F_m has a primitive prime divisor p, and so, $p | n-1$, $n+2$, or $2n+1$. The first two cases imply $s = 1, 2$, and so $p \mid 2n+1$, which can only happen if m is odd and $n \equiv (m-1)/2 \pmod{m}$.

Our Goal

- Bilu, Komatsu, Luca, Pizzaro-Madariaga, P.S. (2015): We look at the relation

$$U_m \mid U_{n+k}^s - U_n^s, \tag{3}$$

with positive integers $k,\ m,\ n,\ s$, where **U** is the general Lucas sequence.

The result

- This year, the quintet (actually, a quartet) got together in Johannesburg, South Africa and looked at the general divisibility relation (3) $U_m \mid U_{n+k}^s - U_n^s$ and proved the following result.

Theorem

Let a be a non-zero integer, $b \in \{\pm 1\}$, and k a positive integer. Assume that $(a, b) \notin \{(\pm 1, -1),\ (\pm 2, -1)\}$. Given a positive integer m, let s be the smallest positive integer such that the divisibility $U_m \mid U_{n+k}^s - U_n^s$ holds. Then either $s \in \{1, 2, 4\}$, or

$$m < 20000(sk)^2. \tag{4}$$

Proof that we (almost) all got together

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Proof method – sketch 1

- Recall the Binet formulas for U_n and V_n:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n \qquad \text{for all} \qquad n \geq 0.$$

- We assume that $m \geq 10000k$. Since $U_{n+4m} \equiv U_n \pmod{U_m}$ ($n \geq 0$, $m \geq 2$), we may assume that $n \leq 4m$. We split U_m into various factors, as follows:

$$U_{n+k}^s - U_n^s = \prod_{d \mid s} \Phi_d(U_{n+k}, U_n),$$

where $\Phi_d(X, Y)$ is the homogenization of the cyclotomic polynomial $\Phi_d(X)$.

Proof method – sketch 2

- We put $s_1 := \mathrm{lcm}[2, s]$, $\mathcal{S} := \{p : p \mid 6s\}$ and

$$\begin{aligned} D &:= (U_m)_{\mathcal{S}};\\ A &:= \gcd(U_m/D, \prod_{d\le 6,\ d\ne 5} \Phi_d(U_{n+k}, U_n));\\ E &:= \gcd(U_m/D, \prod_{\substack{d\mid s_1\\ d=5 \text{ or } d>6}} \Phi_d(U_{n+k}, U_n)). \end{aligned}$$

  Clearly,

$$U_m \mid ADE.$$

- We proceed on bounding A, D, E.

Proof method – sketch 3

- If k is even then

$$\Phi_d(U_{n+k}(-\alpha,-\beta),U_n(-\alpha,-\beta)) = \pm\Phi_d(U_{n+k}(\alpha,\beta),U_n(\alpha,\beta)),$$

while if k is odd, then

$$\begin{aligned}\Phi_d(U_{n+k}(-\alpha,-\beta),U_n(-\alpha,-\beta)) &= \pm\Phi_d(U_{n+k}(\alpha,\beta),-U_n(\alpha,\beta)) \\ &= \pm\Phi_{d^*}(U_{n+k}(\alpha,\beta),U_n(\alpha,\beta))\end{aligned}$$

where

$$d^* = \begin{cases} d & \text{if } 4 \mid d \text{ or } \delta = 1, \\ d/2 & \text{if } 2 \parallel d \text{ and } \delta = -1, \\ 2d & \text{if } 2 \nmid d \text{ and } \delta = -1. \end{cases}$$

Proof method – sketch 4

- Note that $\varphi(d^*) = \varphi(d), \Phi_{d^*}(X) = \pm\Phi_d(\delta X), \Phi_d(X^{-1}) = \pm X^{-\varphi(d)}\Phi_d(X)$, the sign in last identity being "$+$" for $d > 1$ and the sign in the middle identity being "$+$" if $\delta = 1$ or $\min\{d, d^*\} > 1$.
- Note that the sets $\{d \le 6,\ d \ne 5\}$ and $\{d \mid s_1,\ d = 5 \text{ or } d > 6\}$ are closed under the operation $d \mapsto d^*$.
- Hence, $D,\ A,\ E$ do not change if we replace a by $-a$, so we assume that $a > 0$.
- Recall: for any prime number p we put f_p for the index of appearance in the Lucas sequence $\{U_n\}_{n\ge0}$, which is the minimal positive integer k such that $p \mid U_k$.

Bounding D

- First, for $a \geq 1$, if $\mathcal{S}$ is any finite set of primes and m is a positive integer, then

$$(U_m)_{\mathcal{S}} \leq \alpha^2 m \operatorname{lcm}[U_{f_p} : p \in \mathcal{S}].$$

(this follows from Bilu-Hanrot-Voutier's paper from J. Reine Angew. Math. 2001 "*Existence of primitive divisors of Lucas and Lehmer numbers, with an appendix by M. Mignotte*")

- Thus, since $f_p \leq p + 1$,

$$D \leq \alpha^2 m \prod_{p|6s} U_{p+1} < m\alpha^{2+\sum_{p|6s}(p+1)} \leq \alpha^{6s+3+\log m/\log \alpha},$$

where we used the fact that $\sum_{p|t}(p+1) \leq t + 1$, which is easily proved by induction on the number of distinct prime factors of t.

- Note that

$$E \mid \prod_{\substack{\zeta:\zeta^{s_1}=1\\ \zeta\notin\{\pm 1,\pm i,\pm\omega,\pm\omega^2\}}} (U_{n+k} - \zeta U_n), \tag{5}$$

where $\omega := e^{2\pi i/3}$ is a primitive root of unity of order 3.
- Let $K = \mathbb{Q}(e^{2\pi i/s_1}, \alpha)$, which is a number field of degree $d \le 2\phi(s_1) = 2\phi(s)$. Assume that there are ℓ roots of unity ζ participating in the product appearing in the right–hand side of (5), say $\zeta_1, \ldots, \zeta_\ell$. Write

$$\mathcal{E}_i = \gcd(E, U_{n+k} - \zeta_i U_n) \quad \text{for all} \quad i = 1, \ldots, \ell, \tag{6}$$

where $\mathcal{E}_i$ are ideals in $\mathcal{O}_K$. Then relations (5) and (6) tell us that

$$E\mathcal{O}_K \mid \prod_{i=1}^{\ell} \mathcal{E}_i. \tag{7}$$

- We need to bound the norm $|\mathcal{N}_{K/\mathbb{Q}}(\mathcal{E}_i)|$ of $\mathcal{E}_i$ for $i = 1, \ldots, \ell$. First of all, $U_m \in \mathcal{E}_i$. Thus, using Binet formula and $\beta = (-b)\alpha^{-1}$, we get

$$\alpha^m \equiv (-b)^m \alpha^{-m} \pmod{\mathcal{E}_i} \Longleftrightarrow \alpha^{2m} \equiv (-b)^m \pmod{\mathcal{E}_i}. \tag{8}$$

- Further, by Binet and (8) (with $\zeta := \zeta_i$),

$$\alpha^{2n}(\alpha^k - \zeta) - (-b)^{n+k}(\alpha^{-k} - (-b)^k \zeta) \equiv 0 \pmod{\mathcal{E}_i}. \tag{9}$$

- Using a slightly sharper estimate of Φ_v, we obtained that $\alpha^k - \zeta$ and $\mathcal{E}_i$ are coprime, and so, $\alpha^k - \zeta$ is invertible modulo $\mathcal{E}_i$. Now congruence (9) shows that

$$\alpha^{2n+k} \equiv (-b)^n \zeta \left(\frac{\alpha^k - (-b)^k \overline{\zeta}}{\alpha^k - \zeta} \right) \pmod{\mathcal{E}_i}. \tag{10}$$

- We do go through quite a few cases, depending upon the value of $(-b)^n$ and use the following workhorse lemma.

Lemma (Workhorse Lemma)

Let a, b and k be as in the statement of Theorem 1, and assume in addition that $a \geq 1$. Let $v \geq 1$ be an integer and ζ a primitive vth root of unity. Assume that the numbers

$$\alpha \qquad \text{and} \qquad \frac{\alpha^k - (-b)^k \bar{\zeta}}{\alpha^k - \zeta} \tag{11}$$

are multiplicatively dependent. Then we have one of the following options:

(i) $(-b)^k = -1$, $v = 4$;

(ii) $(a, b, k) \in \{(1, 1, 1),\ (2, 1, 1)\}$ and $v \in \{1, 2\}$;

(iii) $(-b)^k = 1$, $v \in \{1, 2\}$;

(iv) $(a, b, k) = (4, -1, 1)$ and $v \in \{4, 6\}$.

Bounding E & A

- The bound we found for E is

$$E \leq \alpha^{22k\phi(s)\sqrt{m}} < \alpha^{22ks\sqrt{m}}.$$

  In the above, we used that $\phi(s) \leq s$.
- A somewhat similar, but slightly more delicate argument shows that

$$A \leq \alpha^{m/2+k+2+132k\sqrt{m}}.$$

Putting these bounds together

- Using $\alpha^{n-2} \le U_n \le \alpha^n$, $n \ge 1$, then

$$\alpha^{m-2} \le U_m \le DAE \le \alpha^{6s+3+\log m/\log\alpha+m/2+k+2+(132k+22ks)\sqrt{m}}.$$

- Since $s \ge 3$, we have $132 + 22s \le 66s$. Since also $1/\log\alpha < 3$, we get

$$m/2 \le (6s + 7 + 3\log m + k) + 66sk\sqrt{m}.$$

- Since $m \ge 10000$, one checks that $6s + 7 + 3\log m + k < ks\sqrt{m}$. Hence,

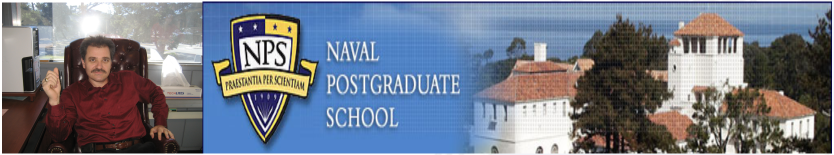
$$m \le 134ks\sqrt{m}, \tag{12}$$

which leads to the desired inequality $m < 20000(sk)^2$.

Further Comments

- One may wonder if one can strengthen our main result in such a way as to include also the instances $s \in \{1, 2, 4\}$ maybe at the cost of eliminating finitely many exceptions in the pairs (a, k).
- The fact that this is not so follows from the formulae:
  - (i) $U_{n+k} - U_n = U_{n+k/2}V_{k/2}$ for all $n \geq 0$ when $b = 1$ and $2\|k$;
  - (ii) $U_{n+k} + U_n = U_{n+k/2}V_{k/2}$ for all $n \geq 0$ when $b = 1$ and $4 \mid k$ or when $b = -1$ and k is even;
  - (iii) $U_{n+k}^2 + U_n^2 = U_{2n+k}U_k$ for all $n \geq 0$ when $b = 1$ and k is odd,

  which can be easily proved using the Binet formulas. Thus, taking $m = n + k/2$ (for k even) and $m = 2n + k$ for k odd and $b = 1$, we get that divisibility $U_m \mid U_{n+k}^s - U_n^s$ always holds with some $s \in \{1, 2, 4\}$.
- Note the "near-miss" $U_{4n+2} \mid 4(U_{n+1}^6 - U_n^6)$ for all $n \geq 0$ if $(a, b, k) = (4, -1, 1)$.



Thank you for your attention!

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