Ekedahl-Oort Stratifications of Unitary Shimura varieties

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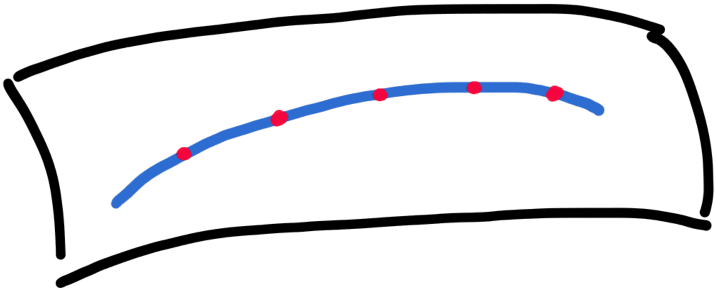


Unitary Shimura varieties

Let K be a quadratic imaginary extension of $\mathbb{Q}$ in which p is unramified. A unitary Shimura variety $\mathcal{M}$ in characteristic $p > 0$ of signature (m_1, m_2) is a moduli space of abelian varieties where:

- $A/\bar{\mathbb{F}}_p$ is an abelian variety of dimension $m_1 + m_2 = g$,
- $\mathcal{O}_K \hookrightarrow \mathrm{End}\,(A)$
- λ is a prime-to-p polarization of A,
- η is a C^p level structure,

Furthermore, under the decomposition of $\mathrm{Lie}(A)$ as an $\mathcal{O}_K \otimes \bar{\mathbb{F}}_p = (\bar{\mathbb{F}}_p)_1 \oplus (\bar{\mathbb{F}}_p)_2$-module, $\mathrm{Lie}(A) = L_1 \oplus L_2$, $\mathrm{rank}(L_i) = m_i$.

Ekedahl-Oort (E-O) Stratification

There exists a poset ${}^JW \subseteq S_g \times S_g$ depending on (m_1, m_2) such that

$${}^JW \longleftrightarrow \left\{ \begin{array}{c} p\text{-torsion group schemes with} \\ \text{extra structure up-to-isomorphism} \end{array} \right\}$$

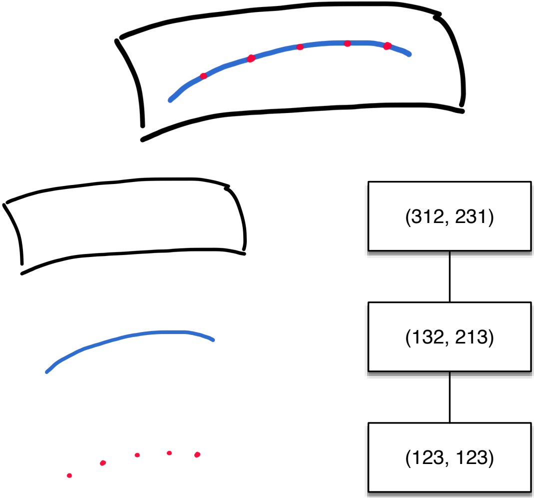
The extra structure is induced by the polarization and endomorphism structure coming from the moduli problem.

Then for $w \in {}^JW$, the subscheme coming from

$$V^w = \{\underline{A} \in \mathcal{M}(\bar{\mathbb{F}}_p) \mid \text{E-O}(\underline{A}) = w\}$$

is the E-O stratum of w.

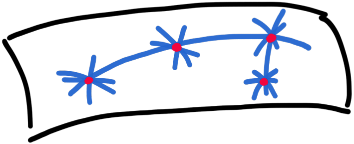
$\mathrm{GU}(2,1)$ p split in K

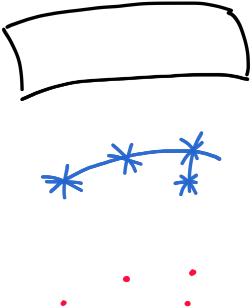


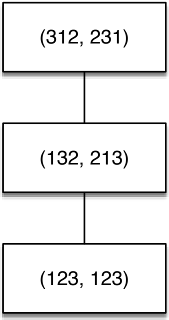
$$312 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$231 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

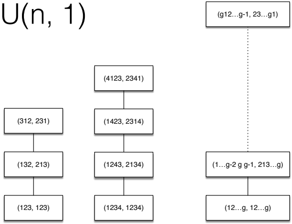
$\mathrm{GU}(2,1)$ p inert in K



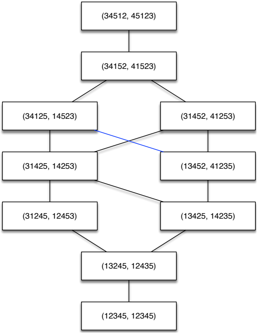




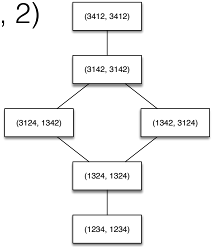
GU(n, 1)



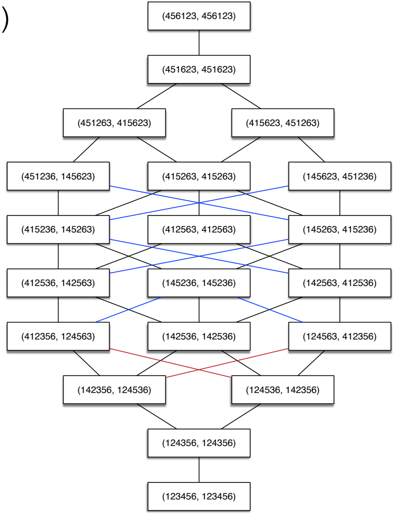
GU(3, 2)



GU(2, 2)



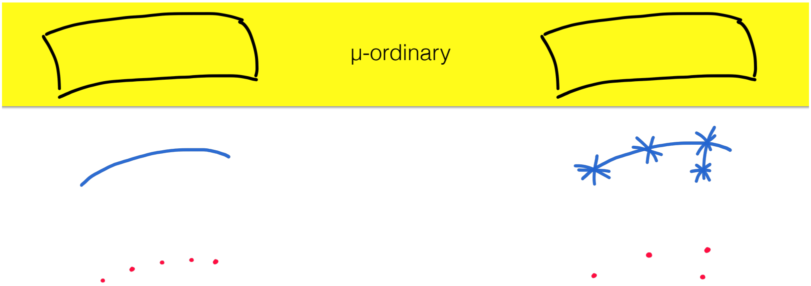
GU(3, 3)

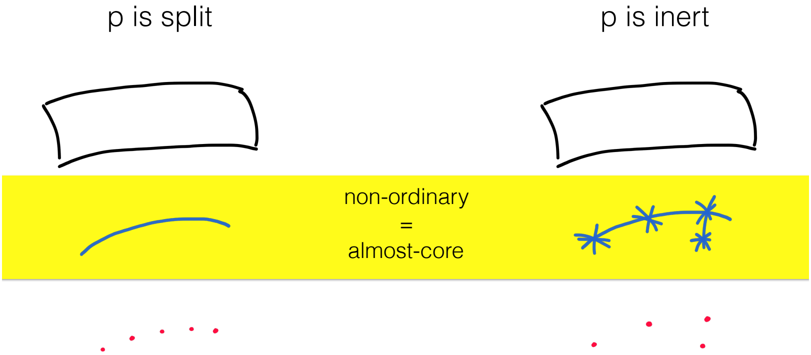

(456123, 456123)
(451623, 451623)
(451263, 415623)
(415623, 451263)
(451236, 145623)
(415263, 415263)
(145623, 451236)
(415236, 145263)
(412563, 412563)
(145263, 415236)
(412536, 142563)
(145236, 145236)
(142563, 412536)
(412356, 124563)
(142536, 142536)
(124563, 412356)
(142356, 124536)
(124536, 142356)
(124356, 124356)
(123456, 123456)

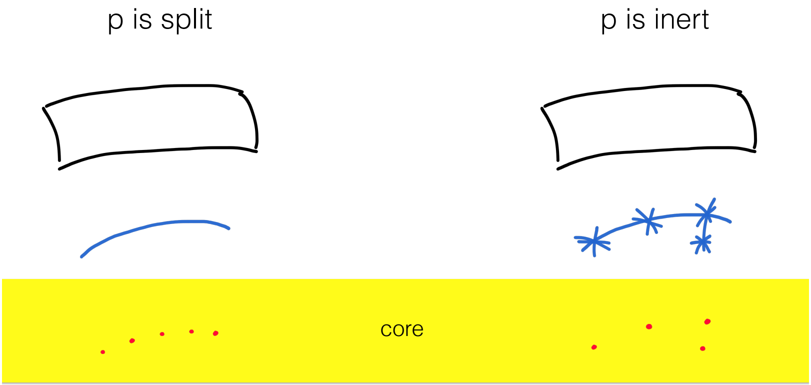
Theorem (W.)

- *The number of E-O strata for* $\mathrm{GU}(m_1, m_2)$ *is* $\binom{g}{m_i}$.
- *The number of strata of a given dimension is given by a partition function.*
- *There is a vertical line of symmetry of the diagram of E-O strata when* $m_1 = m_2$.
- *There exist unique* $0, 1, m_1 m_2 - 1$ *and* $m_1 m_2$*-dimensional E-O strata, the core, almost-core, non-ordinary and* μ*-ordinary strata.*

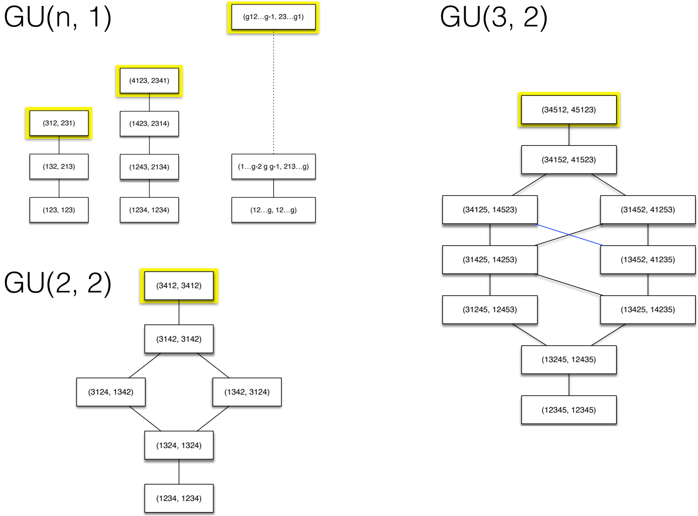
p is split p is inert




p is split
p is inert
non-ordinary
=
almost-core

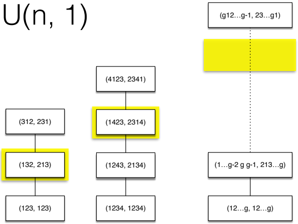

p is split
p is inert
core

μ-ordinary

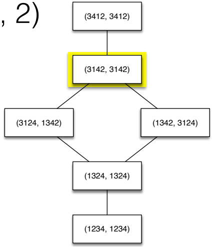

GU(n, 1)
(312, 231)
(132, 213)
(123, 123)
(4123, 2341)
(1423, 2314)
(1243, 2134)
(1234, 1234)
(g12…g-1, 23…g1)
(1…g-2 g g-1, 213…g)
(12…g, 12…g)
GU(2, 2)
(3412, 3412)
(3142, 3142)
(3124, 1342)
(1342, 3124)
(1324, 1324)
(1234, 1234)
GU(3, 2)
(34512, 45123)
(34152, 41523)
(34125, 14523)
(31452, 41253)
(31425, 14253)
(13452, 41235)
(31245, 12453)
(13425, 14235)
(13245, 12435)
(12345, 12345)

Non-ordinary

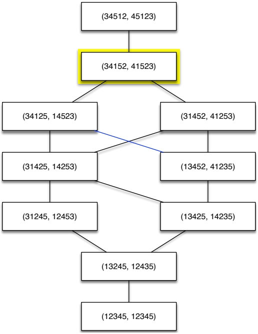
GU(n, 1)



GU(2, 2)

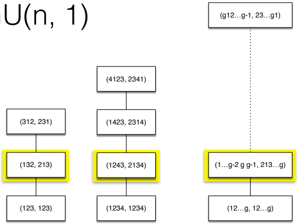


GU(3, 2)

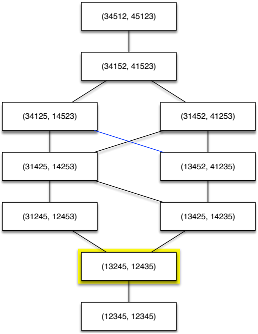


Almost-core

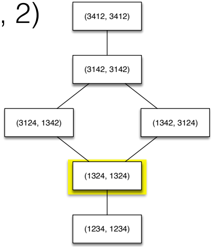
GU(n, 1)



GU(3, 2)

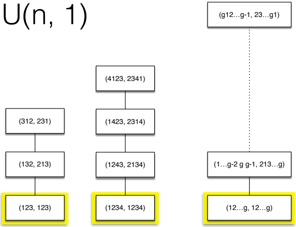


GU(2, 2)

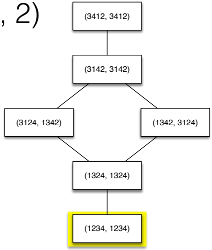


Core

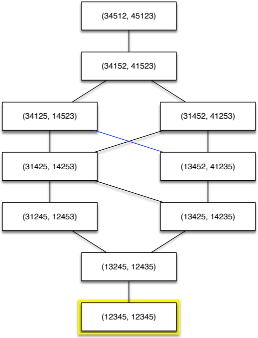
GU(n, 1)



GU(2, 2)



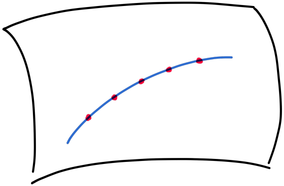
GU(3, 2)

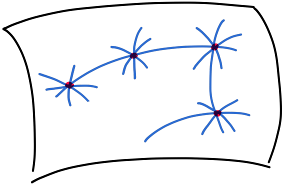


There is a way to go from an element $w \in {}^J W$ to a basis for a Dieudonné module $\mathcal{D}$ with the E-O stratum w. We use these to:

- determine the a-number, f-number, and minimal power of F that kills $\mathcal{D}$ from (m_1, m_2),
- give models for the p-torsion of the μ-ordinary, non-ordinary, almost-core and core E-O strata.

p is split		$a(w)$	$f(w)$	$\mathrm{minF}(w)$
Core		$2m_2$	0	$\lceil \frac{g}{m_2} \rceil$
Almost-core	$m_1 = m_2 = m$	$2m-2$	0	3
	$m_1 - m_2 \geq 1,\ m_2 = 1$	$2m_2 = 2$	1	—
	$m_1 - m_2 \geq 1,\ m_2 > 1,\ m_2 \mid m_1$	$2m_2$	0	$\frac{g}{m_2} + 1$
	$m_1 - m_2 \geq 1,\ m_2 > 1,\ m_2 \nmid m_1$	$2m_2$	0	$\lceil \frac{g}{m_2} \rceil$
Non-ordinary		2	$g-2$	—
μ-ordinary		0	g	—





p is inert		$a(w)$	$f(w)$	$\mathrm{minF}(w)$
Core		g	0	2
Almost-core	$m_1 \geq m_2 > 1$	$g-2$	0	3
	$m_1 > m_2 = 1$	$g-2$	0	4
Non-ordinary	$m_1 = m_2 = m$	2	$2m-2$	—
	$m_1 - m_2 = 1, m_2 = 1$	$m_1 - m_2$	$2m_2 - 2 = 0$	4
	$m_1 - m_2 = 1, m_2 > 1$	$m_1 - m_2$	$2m_2 - 2$	—
	$m_1 - m_2 > 1$	$m_1 - m_2$	$2m_2 - 2$	—
μ-ordinary		$m_1 - m_2$	$2m_2$	—

p is split		$a(w)$	$f(w)$	$\mathrm{minF}(w)$
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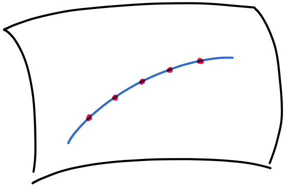
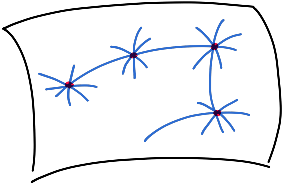
Almost-core	$m_1 = m_2 = m$	$2m-2$	0	3
	$m_1 - m_2 \geq 1,\ m_2 = 1$	$2m_2 = 2$	1	—
	$m_1 - m_2 \geq 1,\ m_2 > 1,\ m_2 \mid m_1$	$2m_2$	0	$\frac{g}{m_2} + 1$
	$m_1 - m_2 \geq 1,\ m_2 > 1,\ m_2 \nmid m_1$	$2m_2$	0	$\lceil \frac{g}{m_2} \rceil$
Non-ordinary		2	$g-2$	—
μ-ordinary		0	g	—





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	$m_1 > m_2 = 1$	$g-2$	0	4
Non-ordinary	$m_1 = m_2 = m$	2	$2m-2$	—
	$m_1 - m_2 = 1, m_2 = 1$	$m_1 - m_2$	$2m_2 - 2 = 0$	4
	$m_1 - m_2 = 1, m_2 > 1$	$m_1 - m_2$	$2m_2 - 2$	—
	$m_1 - m_2 > 1$	$m_1 - m_2$	$2m_2 - 2$	—
μ-ordinary		$m_1 - m_2$	$2m_2$	—

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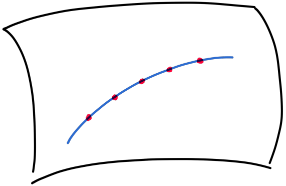
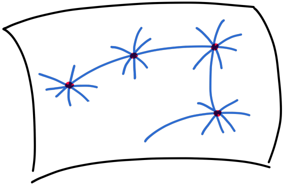
Almost-core	$m_1 = m_2 = m$	$2m-2$	0	3
	$m_1 - m_2 \geq 1,\ m_2 = 1$	$2m_2 = 2$	1	—
	$m_1 - m_2 \geq 1,\ m_2 > 1,\ m_2 \mid m_1$	$2m_2$	0	$\frac{g}{m_2} + 1$
	$m_1 - m_2 \geq 1,\ m_2 > 1,\ m_2 \nmid m_1$	$2m_2$	0	$\lceil \frac{g}{m_2} \rceil$
Non-ordinary		2	$g-2$	—
μ-ordinary		0	g	—





p is inert		$a(w)$	$f(w)$	$\mathrm{minF}(w)$
Core		g	0	2
Almost-core	$m_1 \geq m_2 > 1$	$g-2$	0	3
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Non-ordinary	$m_1 = m_2 = m$	2	$2m-2$	—
	$m_1 - m_2 = 1, m_2 = 1$	$m_1 - m_2$	$2m_2 - 2 = 0$	4
	$m_1 - m_2 = 1, m_2 > 1$	$m_1 - m_2$	$2m_2 - 2$	—
	$m_1 - m_2 > 1$	$m_1 - m_2$	$2m_2 - 2$	—
μ-ordinary		$m_1 - m_2$	$2m_2$	—

Points in the core E-O stratum come from superspecial abelian varieties when p is inert.

Question:

Is there a way to get models for abelian varieties in the core E-O stratum when p is split?

Answer:

Yes! We can construct core points as reductions of certain CM points.

More generally:

Theorem (W.)

Let A *be an abelian variety with CM by* (E, Φ) *where* E/K *is a Galois extension of degree* g*. Then* $A/\bar{\mathbb{F}}_p$ *lies on the Shimura variety associated to* $\mathrm{GU}(m_1, m_2)$*, and its E-O type can be computed from* Φ*.*

Corollary

If $m_1, m_2 \leq 200$ and E is a CM field with totally real subfield E^+ such that

- $E = KE^+$, *p splits in K, p is inert in E^+,*
- E^+ *is cyclic Galois of order $g = m_1 + m_2$,*
- E *has a relative integral basis over K,*

then there is an evenly spaced CM type Φ with respect to m_1, m_2. Furthermore, if there exists an element $\lambda \in \mathcal{D}_{E/\mathbb{Q}}^{-1}$ such that $\bar{\lambda} = -\lambda$, $\Im(\phi(\lambda)) > 0$ for all $\phi \in \Phi$, and $\mathrm{Nm}_{E/\mathbb{Q}}(\lambda)$ is prime-to-p, then the abelian variety associated to (E, Φ) reduce to a point in the core E-O stratum.

Vector bundles over $\mathcal{M}$

Let $\pi : \mathcal{A} \to \mathcal{M}$ be the structure map of the universal abelian scheme $\mathcal{A}$ over $\mathcal{M}$. Then there is an exact sequence of locally free sheaves on $\mathcal{M}$:

$$0 \to \mathbb{E} \to \mathbb{H} \to \mathbb{E}^{\vee} \to 0$$

where $\mathbb{E} = \pi_*(\Omega^1_{\mathcal{A}/\mathcal{M}})$ (the Hodge bundle), $\mathbb{H} = \mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{M})$. The bundles $\mathbb{E}$ and $\mathbb{H}$ split under the $\mathcal{O}_K \otimes \bar{\mathbb{F}}_p$-action, and the ranks of $\mathbb{E}_1, \mathbb{E}_2$ and $\mathbb{H}_1, \mathbb{H}_2$ are determined by the signature (m_1, m_2).

Hodge flags

Let $\mathcal{F} \to \mathcal{M}$ be the space of complete flags of $\mathbb{H}_1$ extending the Hodge filtration:

$$\mathcal{E}_{1,\bullet} : 0 = \mathcal{E}_{1,0} \subsetneq \mathcal{E}_{1,1} \subsetneq \ldots \subsetneq \mathcal{E}_{1,m_1} = \mathbb{E}_1 \subsetneq \ldots \subsetneq \mathcal{E}_{1,g} = \mathbb{H}_1,$$

$$\operatorname{rank}(\mathcal{E}_{1,i}) = i.$$

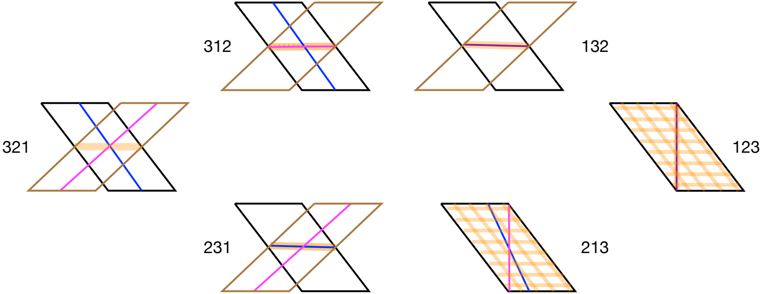
There is a way to construct another flag from $\mathcal{E}_{1,\bullet}$, the *conjugate* flag $\mathcal{D}_{1,\bullet}$.

Stratification on $\mathcal{F}$

$\mathcal{F}$ has a stratification coming from the degeneracy loci of the complete flag $\mathcal{E}_{1,\bullet}$ with respect to $\mathcal{D}_{1,\bullet}$. The closure relations on the strata come from the Bruhat order on S_g. For $w \in S_g$,let $\mathcal{U}^w$ be the stratum of $\mathcal{F}$ corresponding to w.

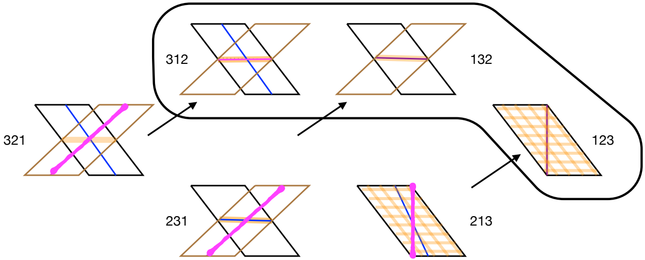
The E-O stratification can be described as the degeneracy loci of the partial flag $0 \subsetneq \mathbb{E}_1 \subsetneq \mathbb{H}_1$ with respect to $\mathcal{D}_{1,\bullet}$.

$\mathcal{F} : GL_3/B \leftrightarrow W$



(The orange indicates the intersection of the flags)

$\mathcal{M} : GL_3/P \leftrightarrow {}^J W$



Theorem (W.)

The map of stratified spaces $\mathcal{F} \to \mathcal{M}$ is isomorphic étale locally to a map of stratified spaces of the form $\mathrm{GL}_g/B \to \mathrm{GL}_g/P$.

Corollary

$\mathcal{F} \to \mathcal{M}$ restricted to $\mathcal{U}^w \to V^w$ where $w \in W$ is a finite, flat, surjective morphism with degree equal to the number of extensions of the canonical flag to a conjugate flag.

Furthermore, the cycle class $[\bar{V}^w]$ can be calculated by pushing forward $[\bar{\mathcal{U}}^w]$. Therefore $[\bar{V}^w]$ lies in the tautological ring.

Thank you!

Hasse-invariants and Hasse-Witt matrices

Question:

What does the deformation theory of Hasse-Witt matrices tell us about the vanishing of the Hasse-invariants and geometry of the moduli space?

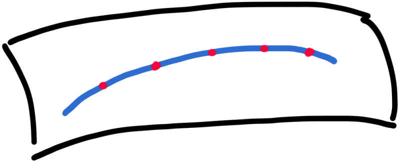
Vanishing of the determinant of the Hasse-Witt matrix

Vanishing of the Hasse-invariant(s)

Closure of the non-ordinary stratum

Theorem (p split in K)

The partial Hasse-invariants H_1, H_2 over $\mathcal{M}$ both vanish to order 1 on the non-ordinary locus, and the intersection of a connected component of $\mathcal{M}$ with the closure of the non-ordinary locus is irreducible.



Theorem (p inert in K)

When $m_1 > m_2 = 1$, the Hasse-invariant vanishes to order 1 on the non-ordinary locus, and the vanishing locus of the Hasse-invariant is locally formally cut-out at the core points by the equation of a Fermat hypersurface:

$$t_1^{p+1} + t_2^{p+1} + \ldots + t_{m_1}^{p+1} = 0.$$