WCNT 2016

Thin Groups and Fractals

Daniel Lautzenheiser and Arthur Baragar

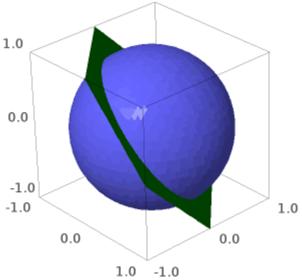
Department of Mathematical Sciences
University of Nevada Las Vegas

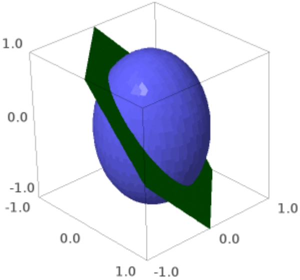
December 14, 2016

Spherical Geometry

$$\mathcal{S} = \{\vec{x} : ||\vec{x}|| = 1\} = \{\vec{x} : \vec{x} \cdot \vec{x} = 1\} = \{\vec{x} : \vec{x}^t I \vec{x} = 1\}$$

Lines: $\mathcal{S} \cap P$
Angles: $\vec{x} \cdot \vec{y} = ||\vec{x}||||\vec{y}|| \cos\theta$
Weighted dot product: $I \mapsto A$, (A positive-definite, symmetric)





Lorentz Product

$$\vec{x} \circ \vec{y} = \vec{x}^t J \vec{y}, \qquad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

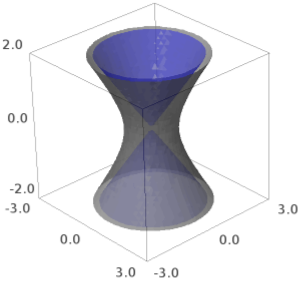
" $\circ$ " is bi-linear and symmetric, but not an inner product.

Light Cone:

$$\mathcal{L}^+ = \{\vec{x} : \vec{x} \circ \vec{x} = 0\}$$

Outer Hyperboloid (1 sheet) :

$$\vec{x} \circ \vec{x} = 1$$



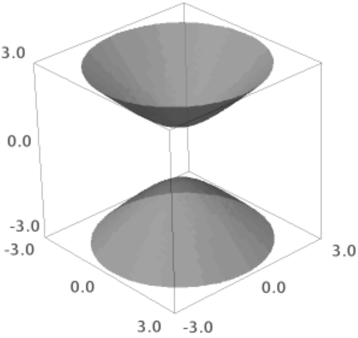
Pseudosphere

Pseudosphere model:

$$\mathcal{V} = \{\vec{x} : \vec{x} \circ \vec{x} = -1\}$$

$$\mathcal{V}^+ = \{\vec{x} = (x, y, z) \in \mathcal{V} : z > 0\}$$

Lines: $P \cap \mathcal{V}^+$

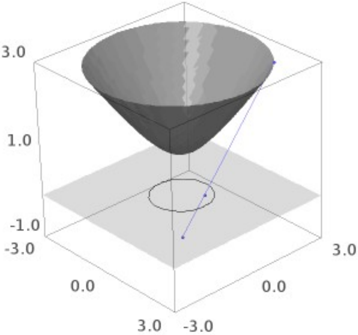


Disk Model

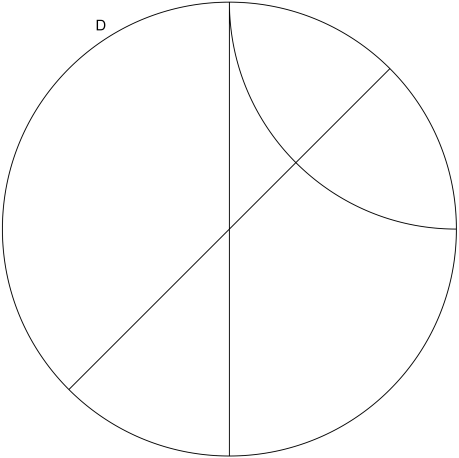
Stereographic Projection:

$$\pi : \mathcal{V}^{+} \longrightarrow D$$

$$D = \{(x, y, 0) : x^2 + y^2 < 1\}$$



Disk Model Cont.

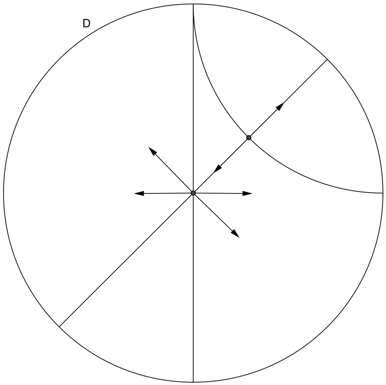


Disk Model: Lines

$\vec{n_1} = (1, 0, 0)$
$\vec{n_2} = (-1, 1, 0)$
$\vec{n_3} = (1, 1, 1)$
Intersect

$$planes : \begin{cases} \vec{x} \circ \vec{n_1} = 0 \\ \vec{x} \circ \vec{n_2} = 0 \\ \vec{x} \circ \vec{n_3} = 0 \end{cases}$$

with $\mathcal{V}^+$ to get lines. Apply π to these lines.



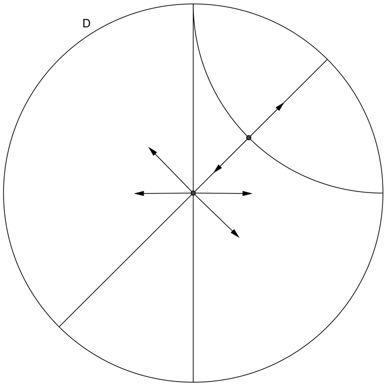
Disk Model: Moving Points

The vectors $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$ define 3 planes. The mapping

$$R_j(\vec{x}) = \vec{x} - 2Proj_{\vec{n}_j}\vec{x} = \vec{x} - 2\frac{\vec{x} \circ \vec{n}_j}{\vec{n}_j \circ \vec{n}_j}\vec{n}_j$$

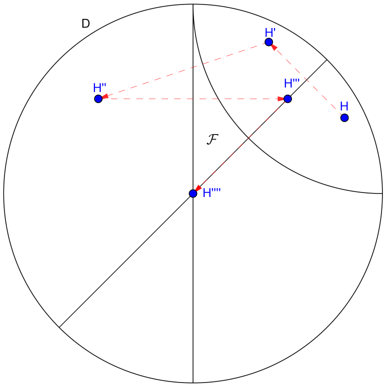
is reflection through the plane defined by $\vec{n}_j$. R_j is an isometry since

$$R_j(\vec{x}) \circ R_j(\vec{y}) = \vec{x} \circ \vec{y}$$



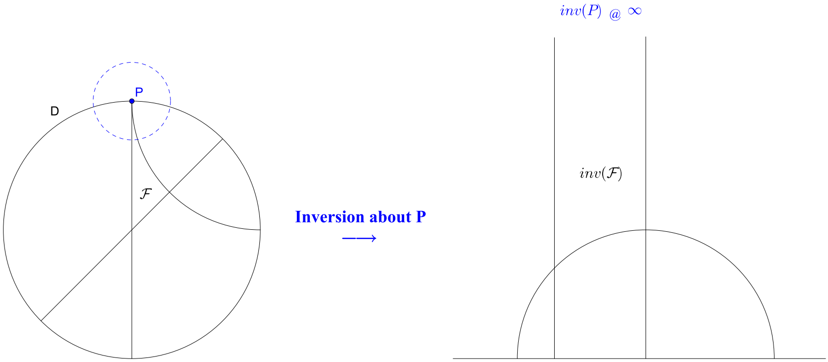
Disk Model: Lattice Points

$$(8,4,9) \mapsto_{R_2} (4,8,9) \mapsto_{R_3} (-2,2,3) \mapsto_{R_1} (2,2,3) \mapsto_{R_3} (0,0,1)$$



- $\mathcal{F}$ is the fundamental domain for the group $\Gamma = \{R_1, R_2, R_3\}$
- Γ acts transitively on the lattice points of $\mathcal{V}^+$
- $\mathcal{F}$ is a Coxeter polyhedron
- Method of descent to $\mathcal{F}$

What does a picture look like after inversion in a circle?

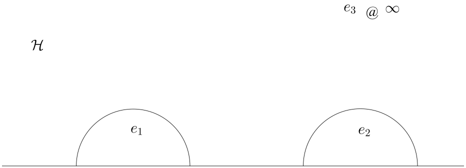


Process: Pictures to/from Lorentz Product

Rules:

1. Choose a basis: $\{e_1, \cdots, e_n\}$
2. Construct $J = [e_i \cdot e_j]$...

Example:

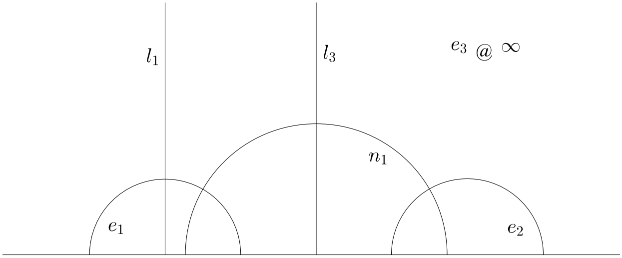


- Think of " $\cdot$ " as " $\circ$ "
- Choose $e_1 \cdot e_1 = e_2 \cdot e_2 = -2$
- Since e_3 is the point at infinity, $e_3 \cdot e_3 = 0$
- $e_1 \cdot e_2 = \pm||e_1||||e_2|| \cos\theta$ *or*... $e_1 \cdot e_2 = \pm||e_1||||e_2|| \cosh\psi$
- $e_1 \cdot e_3 =$ curvature (1/radius)

$$J = \begin{pmatrix} -2 & d & 1 \\ d & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

- J induces "$\circ$"

Fundamental Domain

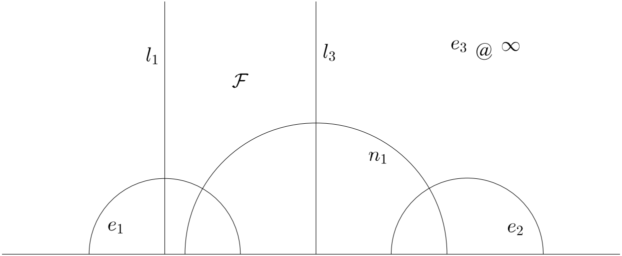


$$\begin{cases} l_1 \cdot e_1 = 0 \iff \vec{l}_1 \circ \vec{e}_1 = 0 \\ l_1 \cdot e_3 = 0 \iff \vec{l}_1 \circ \vec{e}_1 = 0 \end{cases}$$

$$\begin{cases} l_3 \cdot n_1 = 0 \\ l_3 \cdot e_3 = 0 \end{cases}$$

$$\begin{cases} n_1 \cdot e_1 = 0 \\ n_1 \cdot e_2 = 0 \end{cases}$$

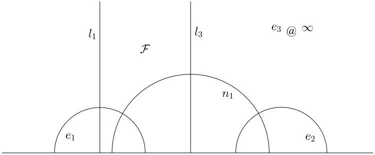
Fundamental Domain



$$\vec{l}_1 = (1, -1, 2+d), \quad \vec{n}_1 = (1, 1, 2-d)$$

$$L_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2d+4 & 1 \end{pmatrix}, N_1 = \begin{pmatrix} 1 & 0 & \frac{2}{d-2} \\ 0 & 1 & \frac{2}{d-2} \\ 0 & 0 & -1 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hyperbolic Translations



$$L_1 L_3 \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$N_1 L_1 \sim \begin{pmatrix} \frac{d+6+4\sqrt{d+2}}{d-2} & 0 & 0 \\ 0 & \frac{d+6+4\sqrt{d+2}}{d-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Since $N_1 L_1$ is an isometry, $\det(N_1 L_1) = 1 =$ *product of eigenvalues*.
- If $d = 1, 3, 4$ then we get fundamental units in the ring of integers $\mathbb{Z}[\sqrt{D}] \subseteq \mathbb{Q}[\sqrt{D}]$, some $D \in \mathbb{N}$.
- *Example:* If $d = 4$, then $Spec(N_1 L_1 L_3) = \{7 \pm 4\sqrt{3}, -1\}$, $Spec(N_1 (L_1 L_3)^2) = \{25 \pm 4\sqrt{39}, -1\}$. (Pell's eqn. for D=3,39)

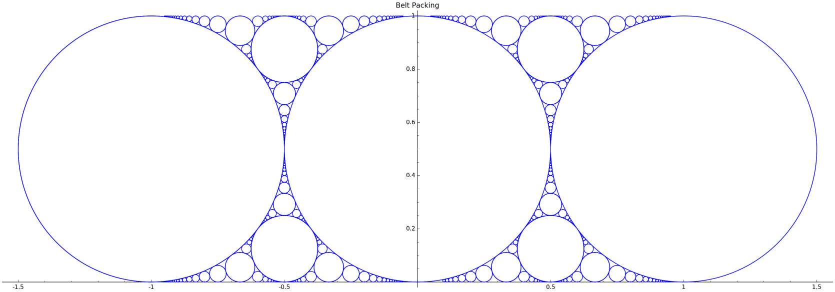
Translations cont.

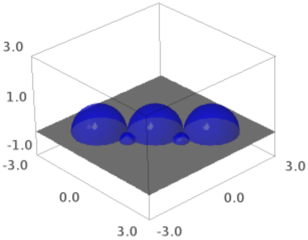
Let $\Lambda = \{N_1(L_1L_3)^k\}_{k=1}^{\infty}$. How big is $\overline{Spec(\Lambda)}$?

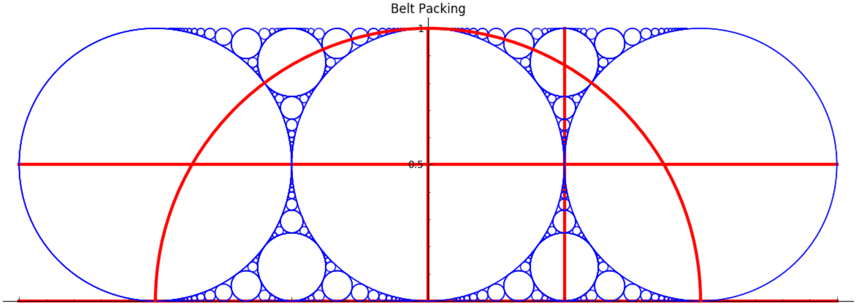
Values of the Discriminant (Within Eigenvalue)

k	d=1	d=3	d=4
1	6	30	3
2	33	105	39
3	78	230	21
4	141	5	3
5	222	70	57
6	321	905	327
7	438	1230	111
8	573	1605	579
9	6	2030	183
10	897	2505	903

Guess (based on first 10,000 cases): $Spec(\Lambda)$ has units in infinitely many D, but misses some values e.g. $D = 7$.


Belt Packing
1
0.8
0.6
0.4
0.2
-1.5
-1
-0.5
0.5
1
1.5

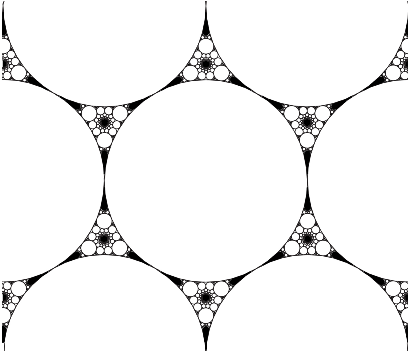

3.0
1.0
-1.0
-3.0
0.0
3.0
-3.0
0.0
3.0

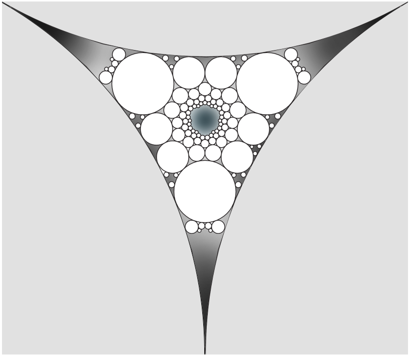


$$J_i = [e_i \cdot e_j] = \begin{pmatrix} -2 & 2 & 2 & 4 \\ 2 & -2 & 2 & 4 \\ 2 & 2 & -2 & 0 \\ 4 & 4 & 0 & 0 \end{pmatrix}, \Gamma = \langle R_1, R_2, R_3, R_4, R_5 \rangle, T = \langle R_1, R_2, R_3, R_4 \rangle$$

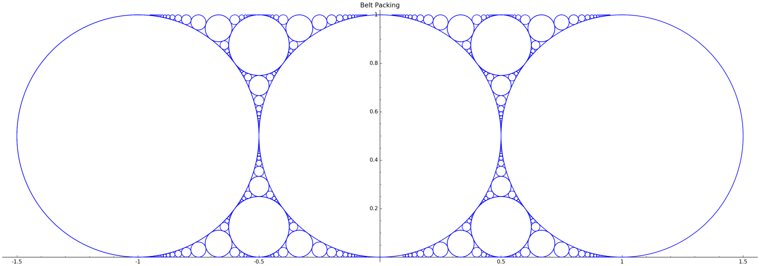
Connection to K3 surfaces

Given a (class of) K3 surface, we may describe a hyperbolic cross section of the ample cone.





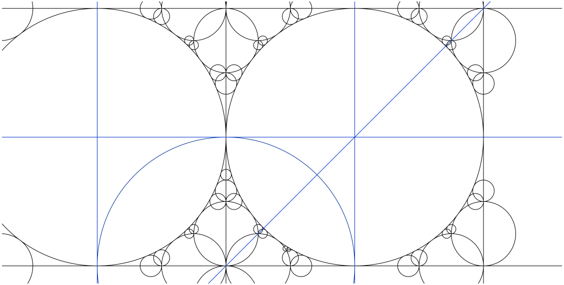
- Each previous J is *even* (even entries on main diagonal), *symmetric*, and has *signature* $(1, n-1)$.
- By [Mor84] and the Hodge Index Theorem, there exists a class of K3 surfaces with intersection matrix $J = [e_i \cdot e_j]$.
- So, these fractals are ample cones of K3 surfaces.
- "Apollonian K3 surfaces"



Further Work

1. Can we modify or generalize the matrix J in the hyperbolic translations case to hit more values of D?
2. Generalized method of descent in "non-Coxeter" cases
3. Using kJ^{-1} as a Lorentz product, some $k \in \mathbb{N}$.

"Dual" belt packing:



References

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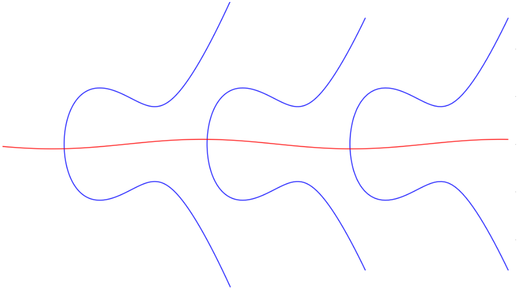
[7] Baragar, Arthur. *Lattice Points on Hyperboloids of One Sheet*. New York Journal of Mathematics. 2013.

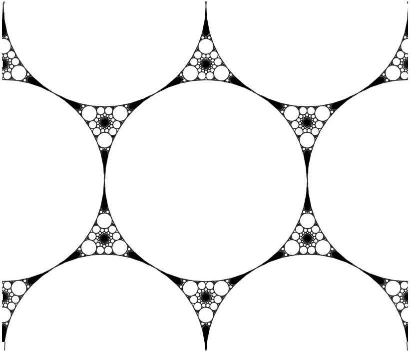
Special thanks to Sarah Glaser for helping me with some of the graphics

Connections to K3 surfaces



- K3 surfaces are examples of elliptic fibrations There are smooth curves (divisor classes) passing through each elliptic curve.
- The isogeny $P \mapsto -P$ sends one smooth curve to another and induces an automorphism of the K3 surface.

Ample Cone Symmetries



Ample cone symmetries $\longleftrightarrow$ arithmetic on elliptic curves

- $O \mapsto O + P$ (horizontal translation fixing ∞)
- $O \mapsto O + Q$ (diagonal translation)
- $P \mapsto -P$ (-1 map through O)

$$J_{amp} = \begin{pmatrix} -2 & 2 & 2 & 4 \\ 2 & -2 & 2 & 4 \\ 2 & 2 & -2 & 4 \\ 4 & 4 & 4 & 0 \end{pmatrix}$$