Formulation and Proof of the Tagiuri Conjecture

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West Coast Number Theory Confernce

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Tagiuri:

- $F_{n+a} F_{n+b} = F_n F_{n+a+b} + (-1)^n F_a F_b$
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- Taiguri: $F_{n+a} F_{n+b} = F_n F_{n+a+b} + (-1)^n F_a F_b$
- Catalan: $F_{n+a} F_{n-a} = F_n F_n + (-1)^{n+a+1} F_a F_a$
- Cassini: $F_{n+1} F_{n-1} = F_n{}^2 + (-1)^n$

Intro: Tagiuri Identities (Caen)

- 1) $P = F_{n+a} F_{n+b} F_{n+c} F_{n+d} F_{n+e} F_{n+f}$
- 2) Trivial Identity: $P = P$
- 3) For each 'pair': (a,b), (c,d), (e,f) on RHS
- 4) Apply Tagiuri
- e.g. $F_{n+a} F_{n+b} \rightarrow F_n F_{n+a+b} + (-1)^n F_a F_b$
- 5) Subsitute: (a,b,c,d,e,f)=(-3,-2,-1,1,2,3)
- Seek Patterns in result

Result

- $F_{n-3}\, F_{n-2}\, F_{n-1}\, F_{n+1}\, F_{n+2}\, F_{n+3}$
- =
- $F_{n-5}\, F_n^{\,4}\, F_{n+5}$ +
- $2F_{n-5}\, F_n - 4F_n^{\,2} - 2F_n\, F_{n+5}$ +
- $(-1)^n\, [F_{n-5}\, F_n^{\,4}\, F_{n+5} + 2F_{n-5}F_n^{\,3}$ –
- $2F_n^{\,3}\, F_{n+5}$ -4]
- Pattern?

Idea Of Theorem

- Patterns in distribution of indices on RHS
- $F_{n-5} F_n^4 F_{n+5}$ +
- $2F_{n-5} F_n - 4F_n^2 - 2F_n F_{n+5}$ +
- $(-1)^n [F_{n-5} F_n^2 F_{n+5} + 2F_{n-5} F_n^3 -$
- $2F_n^3 F_{n+5} -4]$
- n+/- 5 → Count of 4
- n → Count of 16
- Distribution of (4,16,4)

Rules for Counting Indices

- Ignore numerical coefficients
- Ignore $(-1)^n$
- Count multiplicities
- Assume all parenthetical expansions done

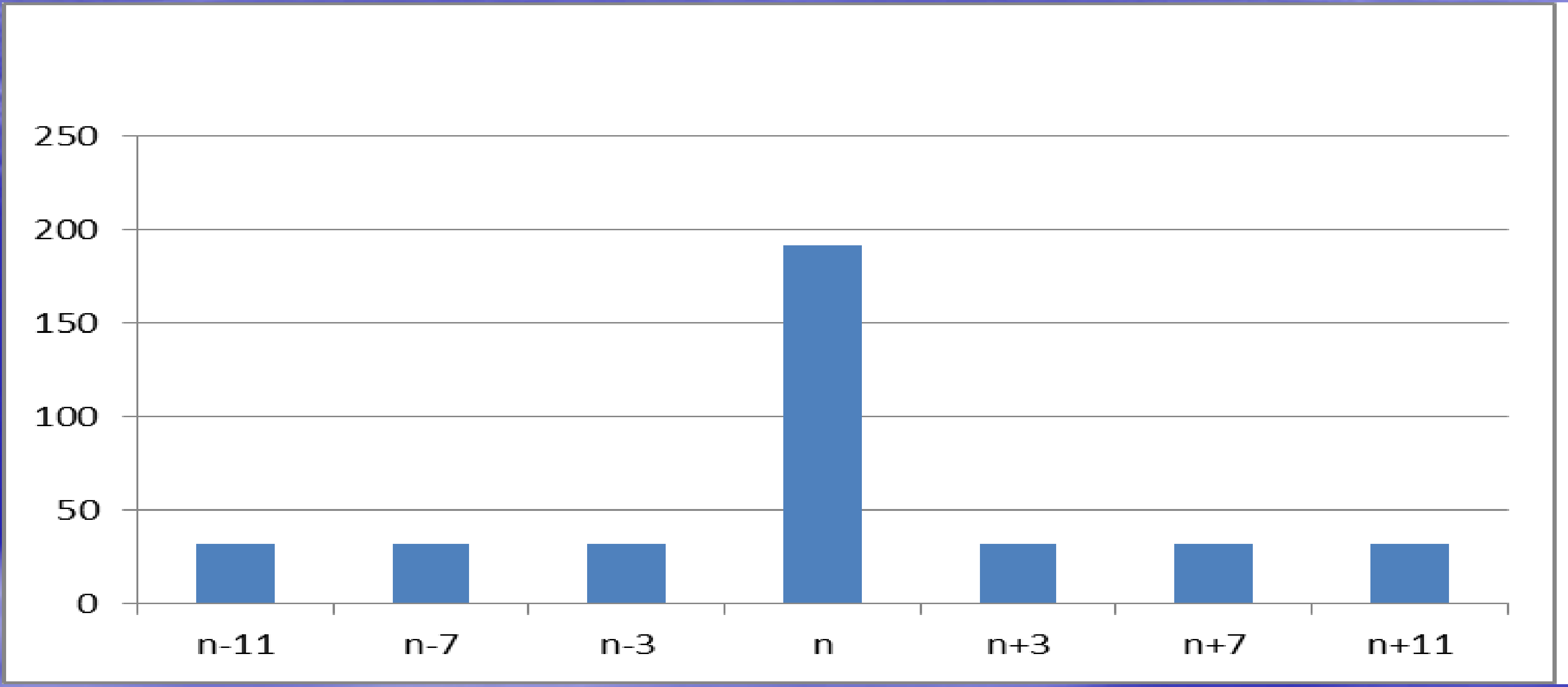
Further Study of Patterns

- Previous slides were for
  - – Initial Products of length 6 with
  - – Tagiuri applied to all consecutive pairs
- Can generalize to initial product of
- Length = $2q$, q=1,2,3,…
- Obtain *family* of Tagiuri generated identities

q=3 and General q

- 0) Let $q=3$
- 1) $P = F_{n+a1}\ldots F_{n+a6}$; General case 2q
- 2) Trivial identity: P = P
- 3) Tagiuri: (1,2; 3,4;….; 2q-1,q)
- RESULTS ON *c*, COUNT of INDICES
- F_{n+a}, q odd, $|a| = 3 \pmod 4$, $|a|<2q \Rightarrow c= 2^{q-1}$
- F_{n+a}, q even, $|a| = 5(4)$, $5 <= |a| <2q \Rightarrow c=2^{q-1}$
- F_n: $c= q2^{q-1}$ (q even); $c=(q+1)\ 2^{q-1}$ (q odd)

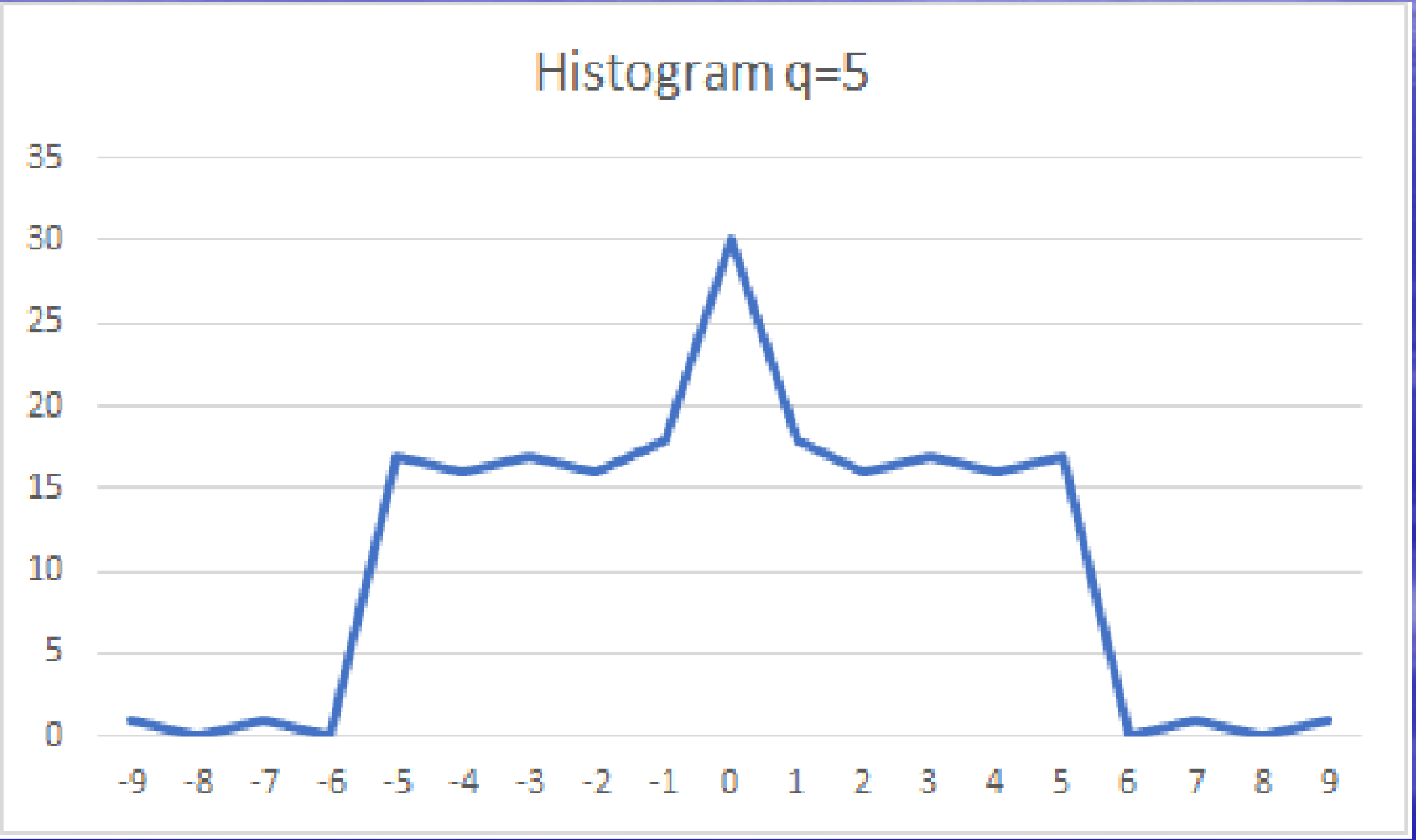
Example 1 (q=6)



Ex 2: Submitted Fib. Quar

- 0) q=2
- 1) $P=F_{n+a}\, F_{n+b}\, F_{n+c}\, F_{n+d}\, F_{n+e}$, 2$q$+1 indices
- 2) Trivial eq: 3P (q+1 positive) - 2P(q neg)
- 3,4) Tagiuri: (1,2),(2,3),(3,4),(4,5),(5,1)
- 5) Substitutions: (1,2,3,4,5)=>(-2,-1,0,1,2)
- RESULTS: F_n weight 6q
- F_{n+e}, Weight 4q-3; F_{n+o} Weight 4q-2
- F_{n+x}, x in $\pm$(q+1,q+2,...,2q-1) Weight 1

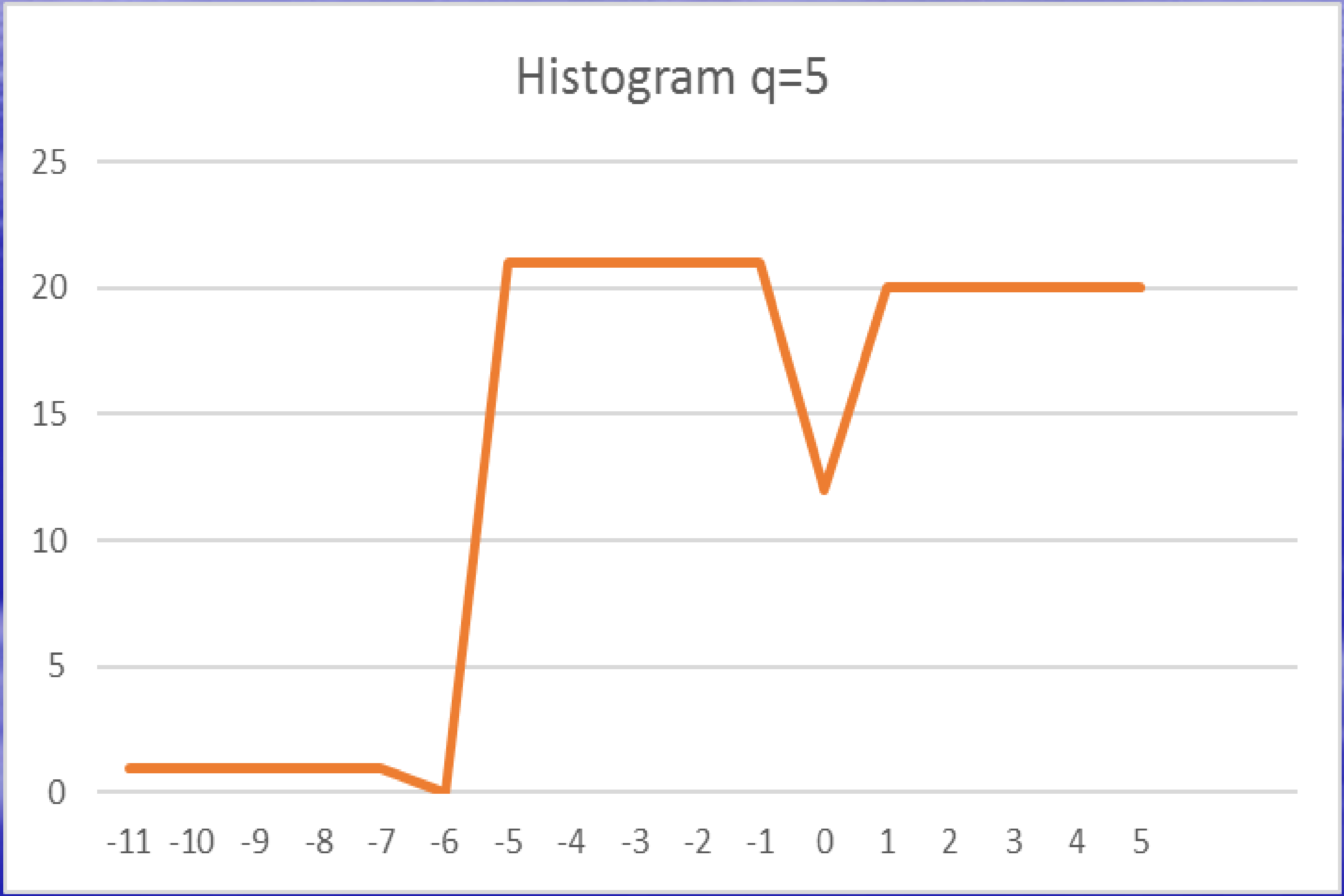
Example 2 (Sub. Fib. Quar)



Example 3 (WCNT 17)

- 0)q=1:1)$P= F_{n+a} F_{n+b} F_{n+c} F_{n+d}$, $2q+2$ in product
- 3) Trivial Identity: $P=P+P-P$ (q+1, +; q, -)
- 4)Tagiuri: (1,2),(1,3), (1,4)
- In general: (1,2),(1,3),…,(1,2q+2)
- --
- RESULTS: n (F_n) → Weight $2q+2$
- n+: 1,2,3…,q→ Weight $4q$
- n-: 1, 2, 3,…,q → Weight $4q+1$
- n-: (q+2),(q+3),…,(2q+1) → Weight 1

Example 3 (WCNT)



PROOFS - Needed Definition#1

- (1,2),(2,3)...(q,1) =>Each indice OCCUR 2ice
- (1,2),(1,3)...(1,2q+2)=>"1" OCCUR Q times
- others OCCUR 1nce

- Definition: Occurrence number of indice *O*(a)
- Definition: Occurence set of family: Collection of *O*(*a*) all *a*

PROOFS - Needed Definitions#2

- *FAMILY* of identities
- (1,2),(1,3), (1,4)
- (1,2),(1,3), (1,4), (1,5), (1,6)
- (1,2), (1,3), (1,4), (1,5),(1,6), (1,7), (1,8)
- FAMILY <==> Substitution String describing any member are prefixes of successor

PROOFS - Needed Definition

- Oops
- (1,2),(1,3),(3,1)
- (1,2),(1,3),(3,4),(4,5),(5,1) <==Not prefix
- ------------ Reinterpret modularly---------
- (1,2),(1,3),(3,4) <== Only 3 indices so 4=1(3)
- (1,2),(1,3),(3,4),(4,5),(5,6)<== 6=1 (mod 5)
- String prefix definition stands with use of modular intepretation.

PROOFS - ISSUES

- Trivial identity: $E(q)$ summands total
- There are $P(q)$ multiplicands per identiy
- Assume 1 Tagiuri application per summand
- I: For each pair $(n+a, n+b)$ in each P,
- get $F_{n+a} F_{n+b} \Rightarrow F_n F_{n+a+b} + (-1)^n F_a F_b$
- So $2E(q)$ occurrences of each index
- II: But each index occurs $O(a)$ times
- Subtract $2O(a)$ from $2E(q)$
- III: Add “1” for each possible “a+b”

PROOFS - ISSUES

- IV) Suppose 0 in (*a*,*b*). Then
- $F_{n+a} F_{n+b} = F_n F_{n+a+b} + (-1)^n F_a F_b$
- But $F_0 = 0$. So we lose summand,

- So Subtract $O(0)$
- V): Add back if $O(a)$ intersect $O(0)$ non empty
- Why? Because I took off twice and should have taken off only once

PROOFS - THEOREM

- *Consider a Tagiuiri Generated Identity with one substitution per summand in the trivial equation. Let n+a, a<>0, be any index. Then the histogram count, H(a), of n+a equals*
- $H(a) = 2E(q) - 2O(a) - O(0) + \#(◎(a) \cap ◎(o)) + ἐ_a$
- *Where ◎ represents the underlying set, # represents cardinality and* $ἐ_a$ *is the number of occurrences of* a in the sums $a_i + a_j$ where (i,j) are members of the Tagiuri substition instructions.

Main Theorem: Illustrate X3 *q*=2

n-3 n-2 n-1 n+1 n+2 n+3	# n-1 n+1 n+2 n+3
n-3 n-2 *n-1* n+1 n+2 n+3	# n-2 n+1 n+2 n+3
n-3 n-2 n-1 *n+1* n+2 n+3	# n-2 n-1 n+2 n+3
n-3 n-2 n-1 n+1 *n+2* n+3	# n-2 n-1 n+1 n+3
n-3 n-2 n-1 n+1 n+2 *n+3*	# n-2 n-1 n+1 n+2

- Look at *n*-2. THere are 2*q*+1 x 2 = 5x2=10
- But O(2) = 1 so take away 2 (r1), leaving 4*q*
- Ditto for n-1 but get 1 back from F_{n-3} F_{n+2}

Main Theorem illustrated: X3

- Example 3: Review slides 12, 13
- Apply theorem: $E(q) = 2q+1$
- ◎(0) = empty $(a_1, a_2, \ldots a_{2q+1}) = (-q\ldots-1,1,\ldots q)$
- Sums of pairs are 2q-1, 2q-3....,7,5,3,1
- ------------------------- Add 1...........Add 1
- $O(1) = 2q+1$; O(not 1) =1
- H(not 1) = $2E(q)$- $2O$(not 1)+$\acute{\varepsilon}_{not\ 1}$ = *2(2q+1)- 2+* $\acute{\varepsilon}_{not\ 1}$ *= 4q, or 4q+1.*
- *Can be seen on slide 13 (or 12)*

Main Theorem: Illustrate X2 *q*=2

n-2	***n-1***	**n**	**n+1**	**n+2**	**#**			**n**	**n+ 1**	**n+2**
n-2	*n-1*	*n*	n+1	n+2	#	n-2			n+1	n+2
n-2	n-1	*n*	*n+1*	n+2	#	n-2	n-1			n+2
n-2	n-1	n	*n+1*	*n+2*	#	n-2	n-1	n		n+2
n-2	n-1	n	n+1	*n+2*	#		n-1	n	n+1	

- Review slides 10,11
- Look at n-2. (2Q+1)*2 = 5x2 = 10 occurrences
- But O(n-2)=2. So remove 2x2; r1,r5. Left w 6
- Also remove RHS row w Fn Row 2,3: Left 4
- So 2*P(q)=2*(2q+1)-2*2-2 = 4Q-4 as required

Corollary

- With assumptions as in the main theorem. Assume further that the total number of occurrence numbers is bounded by *f, finite,* for the entire family. *Then*
- *The non zero range of the histogram (that is the distinct number of numbers counting indices) is finite and in fact*
- *Bounded by 4f+1.*
- *Corollary 2: if O(a)=c → # in range <=5 ☺☺*