On the constant factor in several related asymptotic estimates

Andreas Weingartner

West Coast Number Theory, December 16–20, 2017

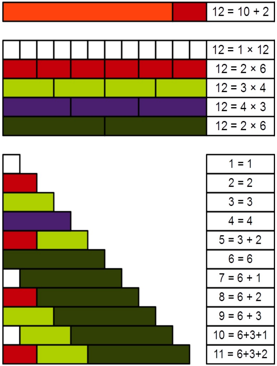
Practical numbers

A positive integer n is called practical if all smaller positive integers can be represented as sums of distinct divisors of n.

Practical numbers

A positive integer n is called practical if all smaller positive integers can be represented as sums of distinct divisors of n.

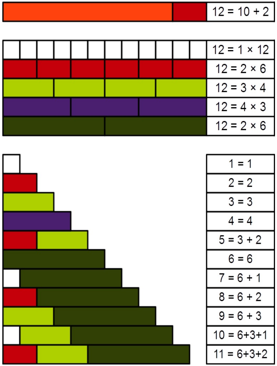
12 is practical:



Practical numbers

A positive integer n is called practical if all smaller positive integers can be represented as sums of distinct divisors of n.

12 is practical:



The sequence of practical numbers:

$$1,\ 2,\ 4,\ 6,\ 8,\ 12,\ 16,\ 18,\ 20,\ 24,\ 28,\ 30,\ 32,\ 36,\ 40, \ldots$$

Characterization of practical numbers

Stewart (1954) and Sierpinski (1955) showed that an integer $n \geq 2$ with prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $p_1 < p_2 < \ldots < p_k$, is practical if and only if

$$p_j \leq 1 + \sigma\left(p_1^{\alpha_1} \cdots p_{j-1}^{\alpha_{j-1}}\right) \qquad (1 \leq j \leq k),$$

where $\sigma(n)$ denotes the sum of the divisors of n.

Characterization of practical numbers

Stewart (1954) and Sierpinski (1955) showed that an integer $n \geq 2$ with prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $p_1 < p_2 < \ldots < p_k$, is practical if and only if

$$p_j \leq 1 + \sigma\left(p_1^{\alpha_1} \cdots p_{j-1}^{\alpha_{j-1}}\right) \qquad (1 \leq j \leq k),$$

where $\sigma(n)$ denotes the sum of the divisors of n.

For example, $364 = 2^2 \cdot 7 \cdot 13$ is practical because

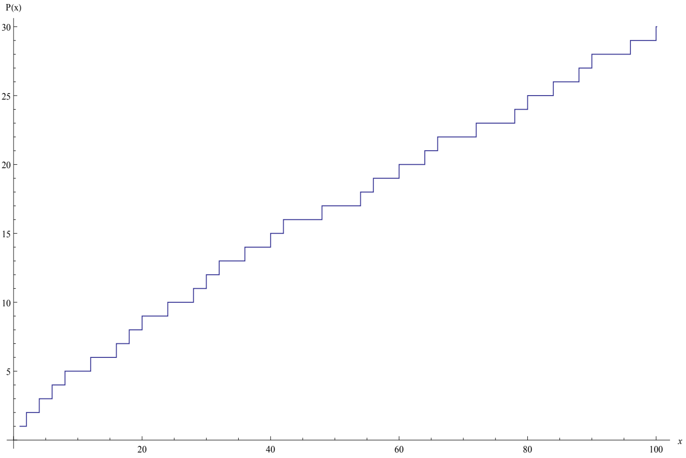
$$2 \leq 1 + \sigma(1) = 2, \quad 7 \leq 1 + \sigma(2^2) = 8, \quad 13 \leq 1 + \sigma(2^2 \cdot 7) = 57.$$

Counting practical numbers up to x

Let $P(x)$ be the number of practical numbers in the interval $[1, x]$.

Counting practical numbers up to x

Let $P(x)$ be the number of practical numbers in the interval $[1, x]$.

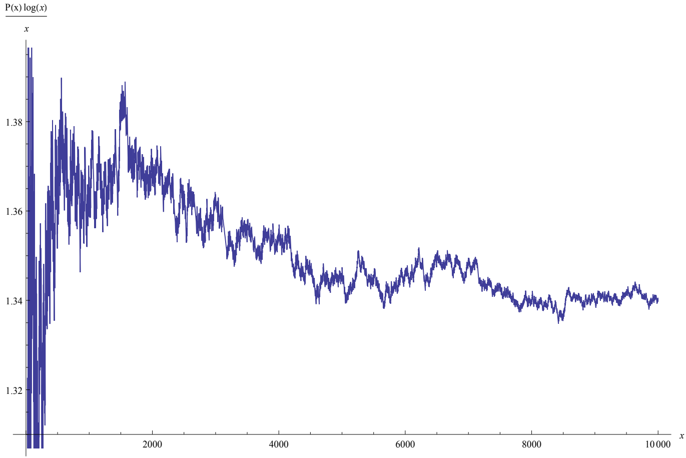


Counting practical numbers up to x

$P(x)$ seems to be about as big as $\dfrac{x}{\log x}$.

Counting practical numbers up to x

$P(x)$ seems to be about as big as $\dfrac{x}{\log x}$. Consider $\dfrac{P(x)}{x/\log x}$:



Counting practical numbers up to x

Srinivasan (1948): $\lim_{x\to\infty} P(x) = \infty$.

Counting practical numbers up to x

Srinivasan (1948): $\lim_{x\to\infty} P(x) = \infty.$

Erdős (1950): $\lim_{x\to\infty} \frac{P(x)}{x} = 0.$

Counting practical numbers up to x

Srinivasan (1948): $\lim_{x\to\infty} P(x) = \infty.$

Erdős (1950): $\lim_{x\to\infty} \frac{P(x)}{x} = 0.$

Saias (1997): $0 < c_1 < \frac{P(x)}{x/\log x} < c_2$

Counting practical numbers up to x

Srinivasan (1948): $\lim_{x\to\infty} P(x) = \infty.$

Erdős (1950): $\lim_{x\to\infty} \frac{P(x)}{x} = 0.$

Saias (1997): $0 < c_1 < \frac{P(x)}{x/\log x} < c_2$

W. (2015): $\lim_{x\to\infty} \frac{P(x)}{x/\log x} = c$ for some constant $c > 0.$

What is the value of c?

W. (2015): $P(x) = \frac{c\,x}{\log x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right)$ for some $c > 0$.

What is the value of c?

W. (2015): $P(x) = \frac{c\,x}{\log x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right)$ for some $c > 0$.

W. (2017):

$$c = \frac{1}{1 - e^{-\gamma}} \sum_{n \in \mathcal{P}} \frac{1}{n} \left(\sum_{p \le \sigma(n)+1} \frac{\log p}{p-1} - \log n \right) \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p}\right)$$

What is the value of c?

W. (2015): $P(x) = \frac{c\,x}{\log x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right)$ for some $c > 0$.

W. (2017):

$$c = \frac{1}{1-e^{-\gamma}} \sum_{n\in\mathcal{P}} \frac{1}{n}\left(\sum_{p\le\sigma(n)+1} \frac{\log p}{p-1} - \log n\right) \prod_{p\le\sigma(n)+1}\left(1-\frac{1}{p}\right)$$

- $\mathcal{P}$ is the set of practical numbers

What is the value of c?

W. (2015): $P(x) = \frac{c\,x}{\log x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right)$ for some $c > 0$.

W. (2017):

$$c = \frac{1}{1-e^{-\gamma}} \sum_{n\in\mathcal{P}} \frac{1}{n}\left(\sum_{p\le\sigma(n)+1} \frac{\log p}{p-1} - \log n\right) \prod_{p\le\sigma(n)+1}\left(1-\frac{1}{p}\right)$$

- $\mathcal{P}$ is the set of practical numbers
- $\sigma(n)$ is the sum of the positive divisors of n

What is the value of c?

W. (2015): $P(x) = \frac{c\,x}{\log x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right)$ for some $c > 0$.

W. (2017):

$$c = \frac{1}{1-e^{-\gamma}} \sum_{n \in \mathcal{P}} \frac{1}{n} \left(\sum_{p \le \sigma(n)+1} \frac{\log p}{p-1} - \log n \right) \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p}\right)$$

- $\mathcal{P}$ is the set of practical numbers
- $\sigma(n)$ is the sum of the positive divisors of n
- p runs over primes

What is the value of c?

W. (2015): $P(x) = \frac{c\,x}{\log x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right)$ for some $c > 0$.

W. (2017):

$$c = \frac{1}{1-e^{-\gamma}} \sum_{n\in\mathcal{P}} \frac{1}{n}\left(\sum_{p\leq\sigma(n)+1} \frac{\log p}{p-1} - \log n\right) \prod_{p\leq\sigma(n)+1}\left(1-\frac{1}{p}\right)$$

- $\mathcal{P}$ is the set of practical numbers
- $\sigma(n)$ is the sum of the positive divisors of n
- p runs over primes
- γ is Euler's constant

What is the value of c?

W. (2015): $P(x) = \frac{c\,x}{\log x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right)$ for some $c > 0$.

W. (2017):

$$c = \frac{1}{1-e^{-\gamma}} \sum_{n \in \mathcal{P}} \frac{1}{n} \left(\sum_{p \le \sigma(n)+1} \frac{\log p}{p-1} - \log n \right) \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p}\right)$$

- $\mathcal{P}$ is the set of practical numbers
- $\sigma(n)$ is the sum of the positive divisors of n
- p runs over primes
- γ is Euler's constant

Corollary: The constant c satisfies $1.311 < c < 1.697$.

What is the value of c?

W. (2015): $P(x) = \frac{c\,x}{\log x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right)$ for some $c > 0$.

W. (2017):

$$c = \frac{1}{1 - e^{-\gamma}} \sum_{n \in \mathcal{P}} \frac{1}{n} \left(\sum_{p \le \sigma(n)+1} \frac{\log p}{p-1} - \log n \right) \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p}\right)$$

- $\mathcal{P}$ is the set of practical numbers
- $\sigma(n)$ is the sum of the positive divisors of n
- p runs over primes
- γ is Euler's constant

Corollary: The constant c satisfies $1.311 < c < 1.697$.
Corollary: Practicals are at least 31% more numerous than primes.

Derivation of the formula for c for practical numbers:

Derivation of the formula for c for practical numbers:

Functional equation from reordering natural numbers:

$$\sum_{m \geq 1} \frac{1}{m^s} = \sum_{n \in \mathcal{P}} \frac{1}{n^s} \prod_{p > \sigma(n)+1} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (\mathrm{Re}(s) > 1)$$

Derivation of the formula for c for practical numbers:

Functional equation from reordering natural numbers:

$$\sum_{m \geq 1} \frac{1}{m^s} = \sum_{n \in \mathcal{P}} \frac{1}{n^s} \prod_{p > \sigma(n)+1} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\mathrm{Re}(s) > 1)$$

Divide both sides by $\zeta(s)$:

$$1 = \sum_{n \in \mathcal{P}} \frac{1}{n^s} \prod_{p \leq \sigma(n)+1} \left(1 - \frac{1}{p^s}\right) \quad (\mathrm{Re}(s) > 1)$$

Derivation of the formula for c for practical numbers:

Functional equation from reordering natural numbers:

$$\sum_{m\geq 1}\frac{1}{m^s}=\sum_{n\in\mathcal{P}}\frac{1}{n^s}\prod_{p>\sigma(n)+1}\left(1-\frac{1}{p^s}\right)^{-1}\quad(\mathrm{Re}(s)>1)$$

Divide both sides by $\zeta(s)$:

$$1=\sum_{n\in\mathcal{P}}\frac{1}{n^s}\prod_{p\leq\sigma(n)+1}\left(1-\frac{1}{p^s}\right)\quad(\mathrm{Re}(s)>1)$$

Differentiate with respect to s:

$$0=\sum_{n\in\mathcal{P}}\frac{1}{n^s}\left(\sum_{p\leq\sigma(n)+1}\frac{\log p}{p^s-1}-\log n\right)\prod_{p\leq\sigma(n)+1}\left(1-\frac{1}{p^s}\right)\quad(\mathrm{Re}(s)>1)$$

Derivation of the formula for c for practical numbers:

$$0 = \sum_{n \in \mathcal{P}} \frac{1}{n^s} \left(\sum_{p \le \sigma(n)+1} \frac{\log p}{p^s - 1} - \log n \right) \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p^s} \right) \quad (\mathrm{Re}(s) > 1)$$

Derivation of the formula for c for practical numbers:

$$0 = \sum_{n\in\mathcal{P}} \frac{1}{n^s} \left(\sum_{p\le\sigma(n)+1} \frac{\log p}{p^s - 1} - \log n \right) \prod_{p\le\sigma(n)+1} \left(1 - \frac{1}{p^s}\right) \quad (\mathrm{Re}(s) > 1)$$

With $s = 1 + 1/\log^2 N$ and $N \to \infty$, the contribution from $n \le N$ is

$$o(1) + \sum_{n\in\mathcal{P}} \frac{1}{n} \left(\sum_{p\le\sigma(n)+1} \frac{\log p}{p - 1} - \log n \right) \prod_{p\le\sigma(n)+1} \left(1 - \frac{1}{p}\right).$$

Derivation of the formula for c for practical numbers:

$$0 = \sum_{n \in \mathcal{P}} \frac{1}{n^s} \left(\sum_{p \le \sigma(n)+1} \frac{\log p}{p^s - 1} - \log n \right) \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p^s} \right) \quad (\mathrm{Re}(s) > 1)$$

With $s = 1 + 1/\log^2 N$ and $N \to \infty$, the contribution from $n \le N$ is

$$o(1) + \sum_{n \in \mathcal{P}} \frac{1}{n} \left(\sum_{p \le \sigma(n)+1} \frac{\log p}{p - 1} - \log n \right) \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p} \right).$$

As $N \to \infty$, the contribution from $n > N$ is

$$o(1) + \int_N^\infty \frac{c}{y^s \log y} \left(\frac{1 - y^{1-s}}{s - 1} - \log y \right) \frac{e^{-\gamma + \int_0^{(s-1)\log y} (1 - e^{-t}) \frac{dt}{t}}}{\log y} dy$$

$$= o(1) + c(e^{-\gamma} - 1)$$

Variation 1: Integers with dense divisors

Variation 1: Integers with dense divisors

The divisors of 20 are $1, 2, 4, 5, 10, 20$.

Variation 1: Integers with dense divisors

The divisors of 20 are $1, 2, 4, 5, 10, 20$.
Each divisor is *at most twice* the next smaller divisor.

Variation 1: Integers with dense divisors

The divisors of 20 are $1, 2, 4, 5, 10, 20$.
Each divisor is *at most twice* the next smaller divisor.

We say n has *dense* divisors if this is the case.

Variation 1: Integers with dense divisors

The divisors of 20 are $1, 2, 4, 5, 10, 20$.
Each divisor is *at most twice* the next smaller divisor.

We say n has *dense* divisors if this is the case.

Let $D(x)$ be the number of integers up to x with dense divisors.

Variation 1: Integers with dense divisors

The divisors of 20 are $1, 2, 4, 5, 10, 20$.
Each divisor is *at most twice* the next smaller divisor.

We say n has *dense* divisors if this is the case.

Let $D(x)$ be the number of integers up to x with dense divisors.

Tenenbaum (1986): $$\frac{1}{(\log\log x)^{4.21}} < \frac{D(x)}{x/\log x} < \log\log x$$

Variation 1: Integers with dense divisors

The divisors of 20 are $1, 2, 4, 5, 10, 20$.
Each divisor is *at most twice* the next smaller divisor.

We say n has *dense* divisors if this is the case.

Let $D(x)$ be the number of integers up to x with dense divisors.

Tenenbaum (1986): $$\frac{1}{(\log\log x)^{4.21}} < \frac{D(x)}{x/\log x} < \log\log x$$

Saias (1997): $$0 < C_1 < \frac{D(x)}{x/\log x} < C_2$$

Variation 1: Integers with dense divisors

The divisors of 20 are $1, 2, 4, 5, 10, 20$.
Each divisor is *at most twice* the next smaller divisor.

We say n has *dense* divisors if this is the case.

Let $D(x)$ be the number of integers up to x with dense divisors.

Tenenbaum (1986): $$\frac{1}{(\log\log x)^{4.21}} < \frac{D(x)}{x/\log x} < \log\log x$$

Saias (1997): $$0 < C_1 < \frac{D(x)}{x/\log x} < C_2$$

W. (2015): $$\lim_{x\to\infty} \frac{D(x)}{x/\log x} = c_2.$$

What is the value of c_2?

W. (2015): $D(x) = \frac{c_2\, x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)$ for some $c_2 > 0$.

What is the value of c_2?

W. (2015): $D(x) = \frac{c_2\, x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)$ for some $c_2 > 0$.

W. (2017):

$$c_2 = \frac{1}{1 - e^{-\gamma}} \sum_{n \in \mathcal{D}} \frac{1}{n} \left(\sum_{p \le 2n} \frac{\log p}{p-1} - \log n \right) \prod_{p \le 2n} \left(1 - \frac{1}{p}\right)$$

where $\mathcal{D}$ is the set of integers with dense divisors.

What is the value of c_2?

W. (2015): $D(x) = \frac{c_2\, x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)$ for some $c_2 > 0$.

W. (2017):

$$c_2 = \frac{1}{1-e^{-\gamma}} \sum_{n\in\mathcal{D}} \frac{1}{n}\left(\sum_{p\le 2n} \frac{\log p}{p-1} - \log n\right) \prod_{p\le 2n}\left(1-\frac{1}{p}\right)$$

where $\mathcal{D}$ is the set of integers with dense divisors.

Corollary: The constant c_2 is given by $c_2 = 1.2248\ldots$.

What is the value of c_2?

W. (2015): $D(x) = \frac{c_2\, x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)$ for some $c_2 > 0$.

W. (2017):

$$c_2 = \frac{1}{1 - e^{-\gamma}} \sum_{n \in \mathcal{D}} \frac{1}{n} \left(\sum_{p \le 2n} \frac{\log p}{p-1} - \log n \right) \prod_{p \le 2n} \left(1 - \frac{1}{p}\right)$$

where $\mathcal{D}$ is the set of integers with dense divisors.

Corollary: The constant c_2 is given by $c_2 = 1.2248\ldots$.

Corollary: Integers with dense divisors are about 22.5% more numerous than primes.

Variation 2: φ-practical numbers

The integer n is called φ-practical if $X^n - 1 \in \mathbb{Z}[X]$ has a divisor of every degree up to n.

Variation 2: φ-practical numbers

The integer n is called φ-practical if $X^n - 1 \in \mathbb{Z}[X]$ has a divisor of every degree up to n.

Let $P_\varphi(x)$ be the number of φ-practical integers up to x.

Variation 2: φ-practical numbers

The integer n is called φ-practical if $X^n - 1 \in \mathbb{Z}[X]$ has a divisor of every degree up to n.

Let $P_\varphi(x)$ be the number of φ-practical integers up to x.

Thompson (2012): For some constants C_1, C_2,

$$0 < C_1 < \frac{P_\varphi(x)}{x/\log x} < C_2.$$

Variation 2: φ-practical numbers

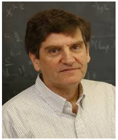
The integer n is called φ-practical if $X^n - 1 \in \mathbb{Z}[X]$ has a divisor of every degree up to n.

Let $P_\varphi(x)$ be the number of φ-practical integers up to x.

Thompson (2012): For some constants C_1, C_2,

$$0 < C_1 < \frac{P_\varphi(x)}{x/\log x} < C_2.$$

Pomerance, Thompson, W. (2016):



$$P_\varphi(x) = \frac{Cx}{\log x}\left\{1 + O\left(\frac{1}{\log x}\right)\right\}$$

for some constant $C > 0$.

Variation 2: φ-practical numbers

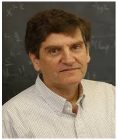
The integer n is called φ-practical if $X^n - 1 \in \mathbb{Z}[X]$ has a divisor of every degree up to n.

Let $P_\varphi(x)$ be the number of φ-practical integers up to x.

Thompson (2012): For some constants C_1, C_2,

$$0 < C_1 < \frac{P_\varphi(x)}{x/\log x} < C_2.$$

Pomerance, Thompson, W. (2016):



$$P_\varphi(x) = \frac{Cx}{\log x}\left\{1 + O\left(\frac{1}{\log x}\right)\right\}$$

for some constant $C > 0$.

W. (2017): The constant $C = \ldots$

Variation 2: φ-practical numbers

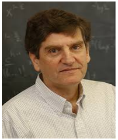
The integer n is called φ-practical if $X^n - 1 \in \mathbb{Z}[X]$ has a divisor of every degree up to n.

Let $P_\varphi(x)$ be the number of φ-practical integers up to x.

Thompson (2012): For some constants C_1, C_2,

$$0 < C_1 < \frac{P_\varphi(x)}{x/\log x} < C_2.$$

Pomerance, Thompson, W. (2016):



$$P_\varphi(x) = \frac{Cx}{\log x}\left\{1 + O\left(\frac{1}{\log x}\right)\right\}$$

for some constant $C > 0$.

W. (2017): The constant $C = \ldots$ and satisfies $0.945 < C < 0.967$.

Variation 3: Polynomials of degree n over $\mathbb{F}_q$ with a divisor of every degree up to n

W. (2016): The proportion of polynomials of degree n over $\mathbb{F}_q$, which have a divisor of every degree up to n, is given by

$$\frac{C_q}{n}\left(1+O\left(\frac{1}{n}\right)\right).$$

Variation 3: Polynomials of degree n over $\mathbb{F}_q$ with a divisor of every degree up to n

W. (2016): The proportion of polynomials of degree n over $\mathbb{F}_q$, which have a divisor of every degree up to n, is given by

$$\frac{C_q}{n}\left(1+O\left(\frac{1}{n}\right)\right).$$

W. (2017): The factor C_q is given by

$$C_q = \frac{1}{1-e^{-\gamma}}\sum_{n\geq 0} f_q(n)\left(\sum_{k=1}^{n+1}\frac{kI_k}{q^k-1} - n\right)\prod_{k=1}^{n+1}\left(1-\frac{1}{q^k}\right)^{I_k},$$

where I_k is the number of monic irreducible polynomials of degree k over $\mathbb{F}_q$ and $f_q(n)$ is the proportion in question.

Variation 3: Polys over $\mathbb{F}_q$ with a divisor of every degree

q	C_q
2	3.400335...
3	2.801735...
4	2.613499...
5	2.523222...
7	2.436571...
8	2.412648...
9	2.394991...

Variation 3: Polys over $\mathbb{F}_q$ with a divisor of every degree

q	C_q
2	3.400335...
3	2.801735...
4	2.613499...
5	2.523222...
7	2.436571...
8	2.412648...
9	2.394991...

For example, the proportion of polynomials of degree n over $\mathbb{F}_2$ having a divisor of every degree up to n is

$$\frac{3.400335...}{n}\left(1+O\left(\frac{1}{n}\right)\right).$$

Variation 3: Polys over $\mathbb{F}_q$ with a divisor of every degree

q	C_q
2	3.400335...
3	2.801735...
4	2.613499...
5	2.523222...
7	2.436571...
8	2.412648...
9	2.394991...

For example, the proportion of polynomials of degree n over $\mathbb{F}_2$ having a divisor of every degree up to n is

$$\frac{3.400335\ldots}{n}\left(1+O\left(\frac{1}{n}\right)\right).$$

We have

$$C_q = \frac{1}{1-e^{-\gamma}} + O\left(\frac{1}{q}\right).$$

Thank You!