Diophantine Equations Counting Non-Ordinary Hyperelliptic Curves

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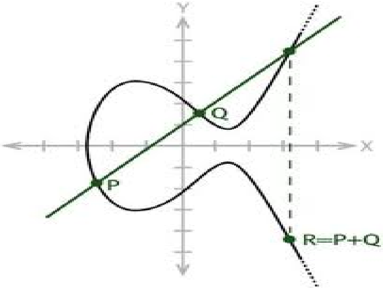
Elliptic Curves

- Let E be an elliptic curve over $k = \overline{\mathbb{F}_q}$, $q = p^n$.
- Then $E : y^2 = x^3 + ax^2 + bx + c = f(x)$ for $p > 2$.
- Algebraic groups law:



- $[\ell] : E \to E$ be mult. by ℓ morphism.
- The ℓ-torsion of E is $E[\ell] = Ker[\ell]$.

Computing the 3 torsion

Recall $E : y^2 = x^3 + ax^2 + bx + c$.

The case $\ell = 3$

$$3Q = Id \Leftrightarrow x(Q) \text{ is a root of the 3-division polynomial}$$

The 3-division polynomial is

$$\Psi_3(X) := 3X^4 + 4aX^3 + 6bX^2 + 12cX + 4ac - b^2$$

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$p = 3$

$\Psi_3(X) = aX^3 + (ac - b^2)$

So $E[3](k) \cong \begin{cases} \mathbb{Z}/3\mathbb{Z} & a \neq 0, \quad \text{E is ordinary} \\ 1 & a = 0, \quad \text{E is supersingular} \end{cases}$

Higher Genus p-ranks of Curves

Let X be an hyperelliptic curve over k of genus g.
$X : y^2 = f(x)$, $deg(f(x)) = 2g + 2$, and $f(x)$ squarefree.
Its Jacobian J_X is a p.p. abelian variety of dimension g.

When $\ell \neq p$:
$J_X[\ell](k) := Ker[\ell] \cong (\mathbb{Z}/\ell)^{2g}$.

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Does each p-rank actually occur? How often is a curve non-ordinary?

The a-number vs. The p-rank

The *p*-rank:

- Let μ_p be the kernel of F on $\mathbb{G}_m$.
- p-rank $f := \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, J_X[p])$

The *a*-number

- Let α_p be the kernel of F on $\mathbb{G}_a$.
- a-number $a := \dim_{\mathbb{F}_p} \mathrm{Hom}(\alpha_p, J_X[p])$

$$\text{Fact: } 0 < a + f \leq g$$

Lemma

X non-ordinary if and only if $a > 0$

Computing the a-number

In General:

$$0 \to H^0(X, \Omega_1) \to H^1_{dR}(X) \to H^1(X, \mathcal{O}) \to 0$$

These spaces have dimensions

$$\dim(H^1_{dR}(X)) = 2g, \quad \dim(H^0(X, \Omega_1)) = g = \dim(H^1(X, \mathcal{O})).$$

Facts:

- $a = dim(\ker(F) \cap \ker(V))$
- $\ker(F) = H^0(X, \Omega_1)$
- $V|_{H^0(X,\Omega_1)}$ is the Cartier operator C
- Thus the a-number is the rank of the kernel of C

Random Matrix Theory

Heuristic: (Cais, Ellenberg, Zureick-Brown)

Rand .Mtx. Thy. predicts the following about (HE) Jacobians:

- a-number strata of (HE) Jacobians are irred. locally closed.
- Heuristic: $P(\text{ordinary}) = \prod_{j=1}^{\infty} \left(1 - q^{1-2j}\right)$

Also;

- They computed f & a of over a billion HE curves.
- The data they got agreed well with their heuristics (though not perfectly).
- Data: $P(\text{ordinary}) = \prod_{j=1}^{(p-1)/2} \left(1 - q^{1-2j}\right)$

In characteristic 3, data suggests there is a $1/q$ chance of being non-ordinary (as $g \to \infty$).

Computing Cartier and the a-number

Calculating the a-number

- $H^0(X, \Omega_1) = Span\{x^i \frac{dx}{y}\}_{i=0}^{g-1}$

The Cartier operator has the properties:

- $C(\omega_1 + \omega_2) = C(\omega_1) + C(\omega_2)$
- $C(z^p \omega) = zC(\omega)$
- $C(dz) = 0$
- $C(dz/z) = dz/z$

Making a Diophantine Problem (in Characteristic 3)

Write

$$f(x)^{\frac{p-1}{2}} = \sum_{j=0}^{p-1} x^j f_j(x^p)$$

then

$$C\left(x^i \frac{dx}{y}\right) = x^{\lfloor i/p \rfloor} f_{i \bmod p}(x) \frac{dx}{y}$$

In characteristic 3, counting non-ordinary hyperelliptic curves is counting 'low height' solutions $Z \in (k[x])^3$ to Diophantine equations of the form

$$Z_0 f_0(x) + Z_1 f_1(x) + Z_2 f_2(x) = 0.$$

The a-number = $\dim_k$(space of low height solutions).

Main Result

Theorem

In characteristic 3, the proportion of curves $y^2 = f(x)$*, with* $\deg(f(x)) = 2g + 2$ *and* $f(x)$ *cube-free with a-number* $a > A$ *is*

$$q^{-2A-1}$$

for any $A \geq g/3$*. No such curves have* $a > (g+2)/3$

Corollary

In characteristic 3, the proportion of non-ordinary curves $y^2 = f(x)$*, with* $\deg(f(x)) = 2g + 2$ *and* $f(x)$ *cube-free is* $1/q$*.*

Thank you