The constant factor in the asymptotic for practical numbers

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West Coast Number Theory, December 16-20, 2019

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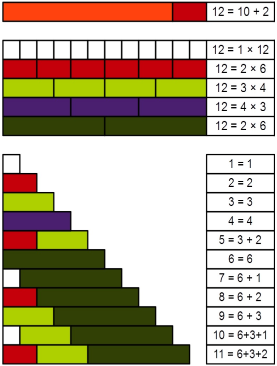

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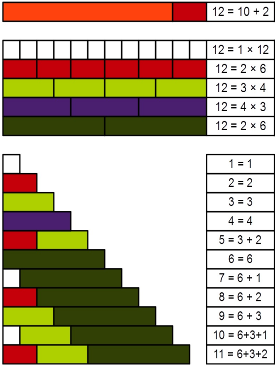

The sequence of practical numbers:

$$1,\ 2,\ 4,\ 6,\ 8,\ 12,\ 16,\ 18,\ 20,\ 24,\ 28,\ 30,\ 32,\ 36,\ 40, \ldots$$

Characterization of practical numbers

Stewart (1954) and Sierpinski (1955) showed that an integer $n \geq 2$ with prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $p_1 < p_2 < \ldots < p_k$, is practical if and only if

$$p_j \leq 1 + \sigma\left(p_1^{\alpha_1} \cdots p_{j-1}^{\alpha_{j-1}}\right) \qquad (1 \leq j \leq k),$$

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For example, $1148 = 2^2 \cdot 7 \cdot 41$ is practical because

$$2 \leq 1 + \sigma(1) = 2, \quad 7 \leq 1 + \sigma(2^2) = 8, \quad 41 \leq 1 + \sigma(2^2 \cdot 7) = 57.$$

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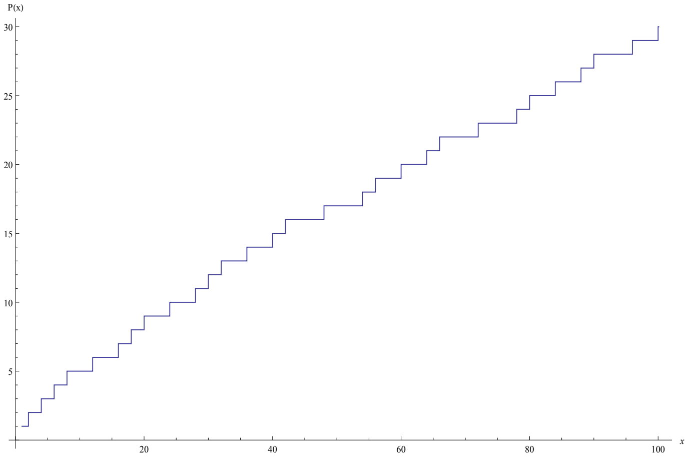

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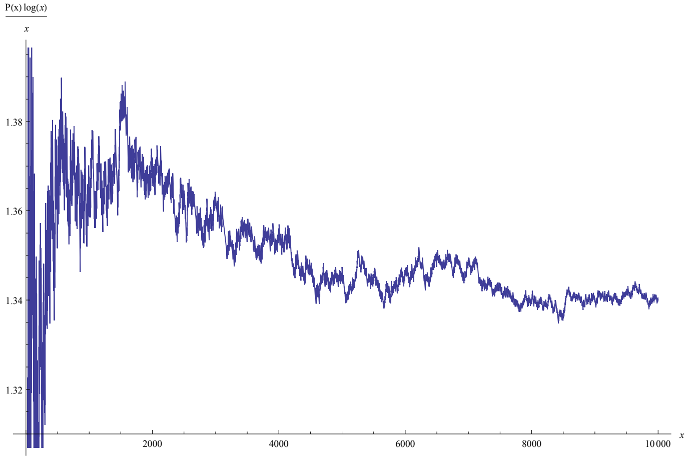

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W. (2015): $\lim_{x\to\infty} \frac{P(x)}{x/\log x} = c$ for some constant $c > 0$.

What is the value of c?

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$$c = \frac{1}{1 - e^{-\gamma}} \sum_{n \in \mathcal{P}} \frac{1}{n} \left(\sum_{p \le \sigma(n)+1} \frac{\log p}{p-1} - \log n \right) \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p}\right)$$

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The practical numbers are between 31% and 70% more numerous than the prime numbers.

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- Most likely, $c = 1.336075$, rounded to six decimal places

The series for c

We need to evaluate the infinite series

$$c(1-e^{-\gamma}) = \sum_{n\in\mathcal{P}} \frac{1}{n}\left(\sum_{p\le\sigma(n)+1} \frac{\log p}{p-1} - \log n\right) \prod_{p\le\sigma(n)+1}\left(1-\frac{1}{p}\right).$$

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So the tail of the series is

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When $N = 2^{31}$, we have $1/\log N \approx 0.05$.

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and

$$\sum_{p\le\sigma(n)+1} \frac{\log p}{p-1} = \log\bigl(\sigma(n)+1\bigr) - \gamma + o(1)$$

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- Partial summation: $|E(x)| < 0.00002174 \quad (x \ge 2^{32})$

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Better idea (2019): Use the fact that $\sigma(n)$ is multiplicative!

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$$\sum_{n\geq 1} \frac{\chi(n)}{n} \log\left(\frac{\sigma(n)}{n}\right) \prod_{p\leq\sigma(n)+1}\left(1-\frac{1}{p}\right)$$

$$= \sum_{n\geq 1} \frac{\chi(n)}{n} \sum_{q^h||n} \log\left(1+\frac{1}{q}+\cdots+\frac{1}{q^h}\right) \prod_{p\leq\sigma(n)+1}\left(1-\frac{1}{p}\right)$$

$$= \sum_{\substack{q\geq 2\\ h\geq 1}} \log\left(\frac{1-1/q^{h+1}}{1-1/q}\right) \sum_{\substack{n\geq 1\\ q^h||n}} \frac{\chi(n)}{n} \prod_{p\leq\sigma(n)+1}\left(1-\frac{1}{p}\right)$$

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The last sum equals

$$\frac{1-1/q}{q^h} \sum_{\substack{n\geq 1\\ \sigma(n)+1\geq q}} \frac{\chi(n)}{n} \prod_{p\leq\sigma(n)+1}\left(1-\frac{1}{p}\right)$$

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where p and q run over primes and

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The series $\sum W_q$ converges rapidly, so the tail of the last series is

$$\sum_{n>N}\frac{\chi(n)}{n}(W+o(1))\prod_{p\leq\sigma(n)+1}\left(1-\frac{1}{p}\right)$$

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- Sort by $\sigma(n)+1$ before finding sums and products, then sort by n after computing these.
- Creating this table takes $N^{1+o(1)}$ steps and $N^{1+o(1)}$ bytes of memory. Calculating c with the use of this table, requires $N^{1+o(1)}$ steps.

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- Without precomputing the products and sums over primes, the algorithm would take $N^{2+o(1)}$ steps.
- Make a table for practical $n \le N$ with rows

$$n, \ \sigma(n)+1, \ \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p}\right), \ \sum_{p \le \sigma(n)+1} \frac{\log p}{p-1}, \ \sum_{q \le \sigma(n)+1} W_q.$$

- Sort by $\sigma(n) + 1$ before finding sums and products, then sort by n after computing these.
- Creating this table takes $N^{1+o(1)}$ steps and $N^{1+o(1)}$ bytes of memory. Calculating c with the use of this table, requires $N^{1+o(1)}$ steps.
- With $N = 2^{31}$ and thirteen hours of computation, we get $1.33607322 < c < 1.33607654$.

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But we don't even know if

$$0.001 \leq \frac{P(x)}{x/\log x} \leq 1000$$

for all $x \geq 20$.

Thank You!