

On Egyptian fractions of length 3

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joint work with

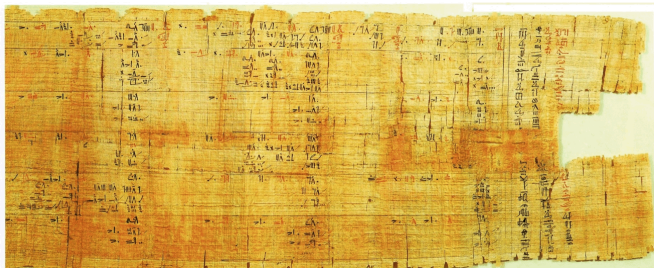
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Egyptian Fractions



EGYPTIAN FRACTIONS

|||| Four

 |||| one-fourth

Write $\frac{2}{5} = \frac{1}{3} + \frac{1}{15} =$   ||||

$3.14 = 3 + \frac{1}{10} + \frac{4}{100} = 3 + \frac{1}{10} + \frac{1}{25} =$ |||   ||||



Egyptian Fractions

Any positive rational a/n can be written as the sum of positive unit fractions

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k}.$$

The above is an example of an Egyptian fraction decomposition of length k .

Here's an example of length 5:

$$\frac{867}{5309} = \frac{1}{7} + \frac{1}{49} + \frac{1}{23650} + \frac{1}{683592739} + \frac{1}{4205691294638106350}$$

Erdős–Straus conjecture

Conjecture

There exist positive integers m_1, m_2, m_3 such that

$$\frac{4}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}.$$

For example:

$$\frac{4}{8675309} = \frac{1}{2168828} + \frac{1}{6271751022618} + \frac{1}{59002291334558621370338268}$$

- It has been verified for $n \leq 10^{14}$.
- The set of exceptions has density 0. (Vaughan 1970)

Since we can't prove it, let's average!

Theorem (Elsholtz–Tao (2013))

Let

$$f(n) = \# \left\{ (m_1, m_2, m_3) \in \mathbb{N}^3 : \frac{4}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\},$$

then

$$x \log^2 x \ll \sum_{p \leq x} f(p) \ll x \log^2 x \log \log x.$$

Functions we want to estimate

$$A_k(n) = \# \left\{ a \in \mathbb{N} : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k} \right\}.$$

$$A_k^*(n) = \# \left\{ a \in \mathbb{N} : \gcd(a, n) = 1, \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k} \right\}.$$

$$f_a(n) = \# \left\{ (m_1, m_2, m_3) \in \mathbb{N}^3 : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\}.$$

$$F(n) = \# \left\{ (a, m_1, m_2, m_3) \in \mathbb{N}^4 : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\}.$$

Length 2

$$A_k(n) = \# \left\{ a \in \mathbb{N} : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k} \right\}.$$

$$A_k^*(n) = \# \left\{ a \in \mathbb{N} : \gcd(a, n) = 1, \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k} \right\}.$$

Theorem (Croot, Dobbs, Friedlander, Hetzel, Pappalardi (2000))

For any $\varepsilon > 0$,

$$A_2^*(n) \ll n^\varepsilon, \quad A_2(n) \ll n^\varepsilon.$$

Furthermore

$$x \log^3 x \ll \sum_{n \leq x} A_2^*(n) \ll x \log^3 x,$$

and

$$x \log^4 x \ll \sum_{n \leq x} A_2(n) \ll x \log^4 x.$$

Length 3, averaging over primes

$$A_k(n) = \# \left\{ a \in \mathbb{N} : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k} \right\}.$$

Theorem (Luca–Pappalardi (2019))

$$x \log^3 x \ll \sum_{p \leq x} A_3(p) \ll x \log^5 x.$$

$A_3(n)$

Theorem (Croot, Dobbs, Friedlander, Hetzel, Pappalardi (2000))

For any $\varepsilon > 0$,

$$A_3(n) \ll n^{\frac{1}{2} + \varepsilon}.$$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

Let $h(n) = C / \log \log n$, where $C = \frac{2 \log(48) \log(\log(6983776800))}{\log(6983776800)} \approx 1.066$.

Then

$$A_3(n) \leq 10n^{\frac{1}{2} + \frac{13}{4}h(n)} \log n.$$

Corollary

For $n \geq 10^{10^{23}}$,

$$A_3(n) \leq \frac{1}{100} n^{\frac{1}{2} + \frac{1}{15}}.$$

$$f_a(n)$$

$$f_a(n) = \# \left\{ (m_1, m_2, m_3) \in \mathbb{N}^3 : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\}.$$

Theorem (Elshotz–Tao (2013))

$$f_4(p) \ll p^{\frac{3}{5}+o(1)}.$$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

For any ρ ,

$$f_a(n) \leq n^\varepsilon \left(\frac{n^{\frac{1}{2}+\frac{\rho}{2}}}{a} + n^{1-\rho} \right).$$

Therefore,

$$f_a(n) \ll \frac{n^{\frac{2}{3}+\varepsilon}}{a^{\frac{2}{3}}}.$$

Explicit $f_a(n)$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

For any ρ , Let $1/3 \leq \rho$, and $n \geq 11000$. Then

$$f_a(n) \leq 6n^{5h(n)} \left(6\sqrt{2} \frac{n^{1/2+\rho/2}}{a} 10^{h(n)} + \frac{3}{2} n^{1-\rho} \log n 6^{h(n)} \right). \quad (1)$$

Corollary

If $n \geq 10^{10^{23}}$, then

$$f_a(n) < \frac{1}{100} n^{\frac{1}{10}} \left(\frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right).$$

$$F(n) = \# \left\{ (a, m_1, m_2, m_3) \in \mathbb{N}^4 : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\}.$$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

Let $\varepsilon > 0$, then

$$F(n) \ll n^{\frac{5}{6} + \varepsilon}.$$

This implies that for large enough n , $F(n) < n$. This suggests the question, what is the largest n such that $F(n) \geq n$.

The first values for which $F(n) < n$ are:

$$F(8821) = 8590, F(11161) = 10270, F(11941) = 10120.$$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

For $n \geq 10^{10^{23}}$.

$$F(n) \leq \frac{1}{10}n$$

Number Theory in the Americas



Parametrization Lemma

Lemma (Luca, Pappalardi)

Consider an Egyptian fraction decomposition of the irreducible fraction a/n :

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \quad \text{with } \gcd(a, n) = 1 \quad (2)$$

Then there exist integers $D_1, D_2, D_3, v_1, v_2, v_3$ with

- (i) $\text{lcm}(D_1, D_2, D_3) \mid n$ and $\gcd(D_1, D_2, D_3) = 1$;*
 - (ii) $av_1v_2v_3 \mid D_1v_1 + D_2v_2 + D_3v_3$ and $\gcd(v_i, D_jv_j) = 1$ when $i \neq j$,*
- and the denominators of the Egyptian fractions are given by*

$$m_i = \frac{n(D_1v_1 + D_2v_2 + D_3v_3)}{aD_iv_i}. \quad (3)$$

Conversely, if conditions (i)–(ii) are fulfilled, then the m_i 's defined via (3) are integers, and denominators of k unit fractions summing to a/n .

Thank you!