Counting elliptic curves with an isogeny of degree three

John Voight
Dartmouth College

Joint work with
Maggie Pizzo and Carl Pomerance

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(for H large), where $d(G)$ is given explicitly.

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G	$\#$ curves $= H^{1/d(G)}$
–	$H^{5/6}$
$\mathbb{Z}/2\mathbb{Z}$	$H^{1/2}$
$\mathbb{Z}/3\mathbb{Z}$	$H^{1/3}$
$\mathbb{Z}/4\mathbb{Z}$	$H^{1/4}$
$\mathbb{Z}/5\mathbb{Z}$	$H^{1/6}$
$\mathbb{Z}/6\mathbb{Z}$	$H^{1/6}$
$\mathbb{Z}/7\mathbb{Z}$	$H^{1/12}$
$\mathbb{Z}/8\mathbb{Z}$	$H^{1/12}$
$\mathbb{Z}/9\mathbb{Z}$	$H^{1/18}$
$\mathbb{Z}/10\mathbb{Z}$	$H^{1/18}$
$\mathbb{Z}/12\mathbb{Z}$	$H^{1/24}$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$H^{1/3}$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$H^{1/6}$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	$H^{1/12}$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	$H^{1/24}$

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The condition at ℓ says we have *overcounted* and need to multiply the result by $1 - \ell^{-10}$; a standard sieve argument then gives

$$\#\{E : \mathrm{ht}(E) \leq H\} = 2^{4/3}3^{-3/2}\zeta(10)^{-1}H^{5/6} + O(H^{1/2}).$$

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Harron–Snowden carried out this strategy for the cases $\#G = 2, 3$:

$$\begin{aligned}\#\{E : \mathrm{ht}(E) \leq H \text{ and } E(\mathbb{Q})_{\mathrm{tors}} \simeq G\} \\ = \frac{\mathrm{area}(R_G)}{\zeta(12/d(G))} H^{1/d(G)} + O(H^{1/e(G)})\end{aligned}$$

for

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Question

Without computing the constant, can one use this method to prove there exists an effectively computable constant for all G?

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In the cases above, is there a (meaningful?) secondary term?

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$$N_{m,\mathrm{cyc}}(H) := \left\{ E : \begin{array}{l} \mathrm{ht}(E) \leq H \text{ and} \\ \text{there exists } \phi\colon E \to E' \text{ cyclic of degree } m \end{array} \right\}.$$

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For $m = 1, 2$, a generator of the kernel of a cyclic m-isogeny is a rational m-torsion point, so we are in the previous case: $N_{m,\mathrm{cyc}}(H) = N_{\mathbb{Z}/m\mathbb{Z}}(H)$ for $m = 1, 2$.

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Example

On $E\colon y^2 = x^3 - 1$, the subgroup $C := \langle \infty, (0, \pm\sqrt{-1}) \rangle$ defines a 3-isogeny $\phi\colon E \to E/C$, where $E/C\colon y^2 = x^3 + 27$ and $\phi(x, y) = (x - 4/x^2, (1 + 8/x^3)y)$.

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Theorem (Pizzo–Pomerance–V)

There exist $c_1, c_2 \in \mathbb{R}$ *such that for* $H \geq 1$,

$$N_{3,\mathrm{cyc}}(H) = \frac{2}{3\sqrt{3}\zeta(6)} H^{1/2} + c_1 H^{1/3} \log H + c_2 H^{1/3} + O(H^{7/24}).$$

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Moreover,

$$\frac{2}{3\sqrt{3}\zeta(6)} = 0.378338\ldots \qquad c_1 = \frac{c_0}{8\pi^2\zeta(4)} = 0.107437\ldots$$

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- The main term of order $H^{1/2}$ counts *just* those elliptic curves with $A = 0$ (having j-invariant 0).
- Matches computations to $H = 10^{25}$, suggesting $c_2 \approx 0.163$.

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So we need to count triples $(A, B, a) \in \mathbb{Z}^3$ satisfying:

(N1) $A \neq 0$ and $\psi_{A,B}(a) = 0$;
(N2) $\left|4A^3\right|, \left|27B^2\right| \leq H$;
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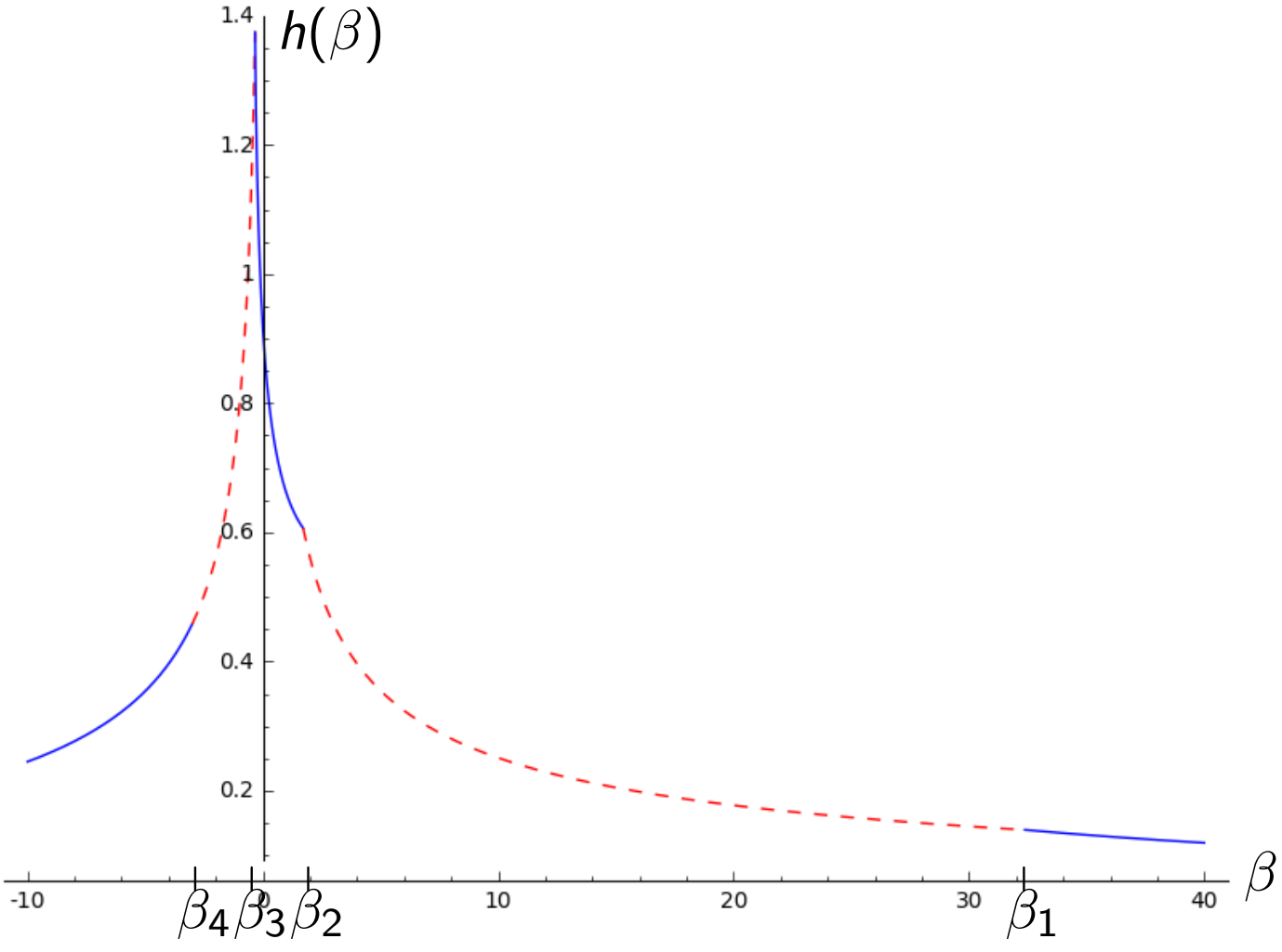
$$12B = \frac{A^2}{a} - 6Aa - 3a^3$$

so it is enough to work with A, a such that $a \mid A^2$, together with conditions at $2, 3$.

Our region



Conclusion (Main result again)

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We hope that our method and the lower-order terms in our result will be useful in understanding counts of rational points on stacky curves (as in recent work of Ellenberg–Satriano–Zureick-Brown).