Western Number Theory Problems, 16 to 18 Dec 2024

for distribution prior to 2025 Asilomar meeting

Edited by Gerry Myerson based on notes by Kjell Wooding

Summary of earlier meetings & problem sets with old (pre 1984) & new numbering.

1967 Berkeley	1968 Berkeley	1969 Asilomar	
1970 Tucson	1971 Asilomar	1972 Claremont	72:01–72:05
1973 Los Angeles	73:01–73:16	1974 Los Angeles	74:01–74:08
1975 Asilomar	75:01–75:23		
1976 San Diego	1–65 i.e., 76:01–76:65		
1977 Los Angeles	101–148 i.e., 77:01–77:48		
1978 Santa Barbara	151–187 i.e., 78:01–78:37		
1979 Asilomar	201–231 i.e., 79:01–79:31		
1980 Tucson	251–268 i.e., 80:01–80:18		
1981 Santa Barbara	301–328 i.e., 81:01–81:28		
1982 San Diego	351–375 i.e., 82:01–82:25		
1983 Asilomar	401–418 i.e., 83:01–83:18		
1984 Asilomar	84:01–84:27	1985 Asilomar	85:01–85:23
1986 Tucson	86:01–86:31	1987 Asilomar	87:01–87:15
1988 Las Vegas	88:01–88:22	1989 Asilomar	89:01–89:32
1990 Asilomar	90:01–90:19	1991 Asilomar	91:01–91:25
1992 Corvallis	92:01–92:19	1993 Asilomar	93:01–93:32
1994 San Diego	94:01–94:27	1995 Asilomar	95:01–95:19
1996 Las Vegas	96:01–96:18	1997 Asilomar	97:01–97:22
1998 San Francisco	98:01–98:14	1999 Asilomar	99:01–99:12
2000 San Diego	000:01–000:15	2001 Asilomar	001:01–001:23
2002 San Francisco	002:01–002:24	2003 Asilomar	003:01–003:08
2004 Las Vegas	004:01–004:17	2005 Asilomar	005:01–005:12
2006 Ensenada	006:01–006:15	2007 Asilomar	007:01–007:15
2008 Fort Collins	008:01–008:15	2009 Asilomar	009:01–009:20
2010 Orem	010:01–010:12	2011 Asilomar	011.01–011.16
2012 Asilomar	012:01–012:17	2013 Asilomar	013.01–013.13
2014 Pacific Grove	014:01–014:11	2015 Pacific Grove	015:01–015:15
2016 Pacific Grove	016:01–016:14	2017 Pacific Grove	017:01–017:21
2018 Chico	018:01–018:19	2019 Asilomar	019:01–019:18
2021 Online	021:01–021:15	2022 Asilomar	022:01–022:21
2023 Henderson	023.01-023.24	2024 Asilomar	024:01–024:18

COMMENTS ON ANY PROBLEM WELCOME AT ANY TIME

48/106 Crimea Road
Marsfield NSW
2122 Australia
gerrymyerson@gmail.com
Australia-2-9877-0133

Problems proposed 16 to 18 December 2024

024:01 (Gerhard Paseman) Let k be a positive integer. What is the probability that if A and B are two multisets of $[n]$ with at most k instances of any element, then $A \subseteq B$?

Solution: (Simon Rubinstein-Salzedo) The answer is $\left(\frac{\binom{k+2}{2}}{(k+1)^2}\right)^n$

024:02 (mathoverflow user Fate Lie via Gerry Myerson) Are there any algebraic irrational x such that $\sum_{n=1}^{\infty}\{n!x\}$ is finite?

Remarks: 1. Problem posted at mathoverflow.net/questions/481321 on 26 October 2024. See also https://mathoverflow.net/questions/484383 posted by GendoTendoLendo, 18 December 2024.

2. For $0 < x < 1$, $x = \sum_{n=1}^{\infty} a_n/n!, a_n\mathbf{Z}$, if $\sum\{n!x\} < \infty$ then $\sum_{n=1}^{\infty} a_n/n < \infty$.

3. (Evan O'Dorney) This is related to normal numbers (factorial base) and seem to be hard.

024:03 (Simon Rubinstein-Salzedo) Suppose a_n is a sequence of integers with infinitely many nonzero terms and $|a_n|$ bounded above. Is $\sum_{n=0}^{\infty} a_n/n!$ always transcendental?

024:04 (Michael Beeson) Suppose $x^2 - dy^2 = 1$ with (x, y) the fundamental (smallest) solution and d squarefree. There will be some squarefree $D > d$ with a larger fundamental solution ($X^2 - DY^2 = 1$ with $X > x$). Can you say anything about how large D might be? In practice it is never as much as twice d. If you want to assume x is a record-breaker, i.e. larger than the fundamental solutions for smaller d, go ahead and assume that too.

Remark: Michael asked a similar question, 022:05.

024:05 (Liam John Lawson) Assume that for each n, a_n is positive and $a_n, a_n + 2, a_n + 3, \ldots, a_n + n$ has no primes. Is it also the case that it contains no prime powers? (Can show for $a_n = n!$, and for sequences involving $\operatorname{lcm}(1, \ldots, n)$)

024:06 (Gerhard Paseman) Let $k > 0$. Let $c > 0$ be the unique integer such that $c^2 < k \leq (c+1)^2$. Now consider the $2c+1$ integers in $[k-c, k+c]$. How large is

$$w(k) = \omega\left(\prod_{i=-c}^{c}(k+i)\right)?$$

Easily $w(k) \leq (2c+1) + \pi(c)$, less obviously $w(k) \leq (2c+1) + \pi(c/\sqrt{2})$. Is there a nice argument to show $w(k) \leq 2c+1$ always?

024:07 (Mathoverflow user Prism, via Gerry Myerson)

The polynomial $p(x, y, z) = z^2(x^2 - y^2z - 1)^2 + z$, where $x, y, z > 0$ integers, has the property that the positive range of p (for positive integers x, y, z) consists of precisely the nonsquare positive numbers. It's easy to show there's no such one-variable polynomial. What about a two-variable polynomial $r(x, y)$? Is there one that "captures" all and only the non-squares?

Remark: Problem posted at mathoverflow.net/questions/432878 on 21 October 2022. Another three-variable example given there is $(x + y + z - 1)^2 - (4x + 2y - 4)$.

024:08 (Ingrid Vukusic) Do there exist infinitely many integers $1 < a < b < c$ such that (a, b, c) and $(a+1, b+1, c+1)$ are both 3-multiplicatively dependent (i.e., multiplicatively dependent, and no sub-pair is multiplicatively dependent).

Remark: See also 022:15.

024:09 (MathOverflow user Benjamin Lang via Gerry Myerson) Do there exist positive integers such that if you add $1, \ldots, a-1$, and the sum of integers from $a+1$ to $b-1$, and the sum of integers from $b+1$ to c, all three sums are the same?

If so, $c > 10^{600,000}$. Peter Taylor improved this to $10^{12,248,800,000}$. Max Alekseyev found an equivalent formulation:

Find squares (besides first two terms) in the linear recurrence sequence,

$$9, 25, 481, 14961, 500889, 16973353, \ldots$$

which can be described equivalently by $t_n = 41t_{n-1} - 246t_{n-2} + 246t_{n-3} - 41t_{n-4} + t_{n-5}$ for $n \geq 5$ and explicit formula

$$t_n = \frac{3}{8}\left((1+\sqrt{2})^{4n} + (1-\sqrt{2})^{4n}\right) + \frac{4-\sqrt{2}}{2}(1+\sqrt{2})^{2n} + \frac{4+\sqrt{2}}{2}(1-\sqrt{2})^{2n} + \frac{17}{4}.$$

Remark: Problem posted at mathoverflow.net/questions/473568 on 20 June 2024.

Solution: A solution, showing that no such positive integers exist, was posted in outline by MathOverflow user Andrew K on 11 December 2025.

024:10 (Andrés Valloud) Define the function $T(k) = 5k + 1$ if k odd, and $k/2$ if k even. What is the limit as n goes to infinity of $T^n(7)$, i.e. does it go to ∞ or is it periodic?

Remarks: 1. One would expect it to go to infinity.

2. In a comment on a deleted answer at https://math.stackexchange.com/questions/14569/the-5n1-problem Gottfried Helms writes that after 936100 iterations the sequence starting with 7 reaches a number of 90900 digits.

024:11 (Stefan Erickson) Let $S(n, k)$ be the Stirling number of the second kind; Let $(n)_k$ be the falling factorial. It is known that $n^n = \sum_{k=0}^{n} S(n, k)(n)_k$. Question: is there a closed form for the sum $\sum_{k=0}^{n} S(n, k)(n)_k 2^{n-k}$?

Remarks: 1. We have the alternate sum $\sum_{k=0}^{n} S(n, k)(n)_k 2^{n-k} = \sum_{k=0}^{n} \binom{n}{k} k^n$

2. (Sungjin Kim) This appears at Project Euler 830.

I answered this question at math.stackexchange.com/questions/355262, which uses a generating function $(e^x + 1)^n$. So, for this question, my answer (still not a closed form) is

$$\left(\frac{d}{dx}\right)^n (e^x + 1)^n \Bigg|_{x=0}$$

024:12 (Math SE User Bryle Morga, via Gerry Myerson) Beginning with a given n, double n and remove any zeroes from the decimal expansion. It is expected that all numbers n will end in a cycle (confirmed up to 10^9) The number 118 generates a cycle of length 432, with peak $234,715,136$ Question: Is that the longest cycle possible?

Remarks: 1. The question was posted to math.stackexchange.com/questions/4895740 on 9 April 2024

2. OEIS A242350 gives the sequence generated by $n = 1$; the end cycle begins on the 471st iteration, and peaks above $11,000,000$.

3. (Gerhard Paseman) What can we say about the equivalence classes in the sequence (i.e. n which culminate in the same cycle?) What happens when we ’invert’ the process / divide by 2?

4. (Michael Beeson) Here is SageMath code to look for such loops. It does duplicate the results stated on Stack Exchange. It’s interesting because you need a hash table (dictionary) for efficient loop detection, which makes a C implementation less straightforward. Stack Exchange says someone searched up to 10^9. This version was updated at 4:48 pm Tuesday; it runs faster than the first version thanks to storing values from previous runs, which we can cut short because they won’t lead anywhere new. It’s doing about thirty million numbers per hour.

[Note: your editor has not learned how to TEXcode blocks, so what follows probably bears little resemblance to actual working code]

“‘ def removezeroes(x): if x ¡ 10: return x; if x % 10 == 0: return removezeroes(x//10); return 10*removezeroes(x//10) + (x % 10) “‘

“‘ oldvals = Dictionary to store elements we’ve already checked lengths = store lengths we find

def dec17(n): seen = Dictionary to store x as key and m as value x = n; m = 1; while m ¡ 1000: if x in seen: firstm = seen[x] len = m - firstm print(f”Sequence starting at n enters a loop of length len at index firstm”) oldvals.update(seen) lengths[n] = len return len; print(m, x) seen[x] = m; x = removezeroes(2*x) if x in oldvals: return; m = m+1; return -1;

def dec17A(start, howmany): maxsofar = 10; for x in range(start,start + howmany): m = dec17(x); if m ¿ maxsofar : maxsofar = m print(x,m) if x print(x) x+= 1 “‘

024:13 (Gerhard Paseman) Let m, r, k be integers which are solutions to $2^k = \sum_{i=0}^{r} \binom{m}{i}$. We care only about interesting solutions, where $2 < 2r < m - 1$. There are two such that are known: $(23, 3, 11)$ and $(90, 2, 12)$. Suppose (m, r, k) is interesting.

Fact1 (P): for every prime p (prime power $q = p^f$) belonging to $[m - r, m]$, one has $2^m \equiv 2^{k+1} \pmod{p}$. As a result, there can't be many primes or prime powers in $[m-r, m]$, because the orders of 2 mod p can't all divide $m - k - 1$.

Fact2 (P): Let $p > r$ be prime (or $q > r$, $q = p^f$) such that $p \mid m - j$, where $0 \le j \le r$. Then $2^k \equiv 2^j \pmod{p}$. Thus orders of 2 mod p with $p \mid m-j$ will alternate in parity with j. (Example: $o_{23}(2) = 11$, $o_{11}(2) = 10$ (dividing 22), $o_7(2) = 3$ (dividing 21, 3 is too small), so orders are odd, then even, then odd.) Thus interesting solutions (m, r, k) correspond to patterns of $o_p(2)$ of prime divisors of consecutive integers from $[m - r, m]$.

Question: What is known about $o_{'n'}(2)$ as n increases, where $o_{'n'}$ means o_p for $p \mid n$ sufficiently large chosen at our discretion? For example, it can be shown that $o_p(2) \neq p-1$ in $[m - r, m]$ for any interesting (m, r, k). Nor can consecutive primes p, t belong to $[m - r, m]$ if $\mathrm{lcm}(o_p(2), o_t(2)) > m$. Nor can consecutive integers $a, a + 1$, if $p \mid a$, $t \mid a + 1$, $p, t > r$ and $o_p(2), o_t(2)$ are both even. Knowing even the sizes of $o_{'n'}(2)$ as n increases could greatly limit the search for interesting solutions (m, r, k).

I am also interested in references to sum of binomial coefficients equaling a power of two.

024:14 (Katie Ahrens) Let $p > 3$ be prime, and define $m_e = \lfloor 2\sqrt{p^e} \rfloor$, $e \ge 5$, e odd. Question: When does $p \mid m_e$? Do there exist p such that, for all e, $p \nmid m_e$? Do there exist p such that $p \mid m_e$ for infinitely many e?

Remarks: 1. This is related to numbers of points on curves over finite fields.

2. Serre answers the second question in the affirmative for $p = 2$.

3. (Anay Aggarwal) If (and only if) there are infinitely many zeroes in odd positions in the base p expansion of $2\sqrt{p}$, the second is true for all p. Hence, if $2\sqrt{p}$ is (weakly) normal (in base p), we may answer the question in the affirmative.

4. See Katie Ahrens, Jon Grantham, On a remark of Serre, https://arxiv.org/abs/2508.11048

024:15 (Alexander Ganahl, via Evan O'Dorney) Take a positive integer n, pick any block of consecutive digits in its base 10 expansion, and multiply or divide their value by 3 (e.g. $7(1)4 \mapsto 734$, $49(66)2 \mapsto 50982$, $8(21)8 \mapsto 8078$). Questions:

1. For which values of a, b can we say a "leads to" b (that is, starting with a, there is a sequence of operations that results in b). E.g. a leads to 1 iff $a \equiv 1, 3, 9, 27 \pmod{40}$)

2. As 10 and 3 vary, what happens to 40?

024:16 (Mathoverflow user Dan via Gerry Myerson)

$$\binom{62}{26}^2 + \binom{62}{27}^2 = \binom{62}{28}^2$$

Does any other row of Pascal's triangle contain a Pythagorean triple?

Remarks: 1. None in the first 30,000 rows.

2. (Florian Luca) no others of the form $\binom{n}{k}^2 + \binom{n}{k+1}^2 = \binom{n}{k+2}^2$.

3. Posted to mathoverflow.net/questions/478688, 12 September 2024.

024:17 (math.stackexchange user Mike Pierce, via Gerry Myerson) Is there a slick way to enumerate ways to write n as a sum of consecutive squares?

Remarks: 1. Posted to math.stackexchange.com/questions/5007665, 5 December 2024.

2. Related entries at the Online Encyclopedia of Integer Sequences:

oeis.org/A130052 tabulates numbers that are sums of consecutive squares in more than one way.

oeis.org/A234304 is sums of consecutive squares in at least three ways. The smallest is $20449 = 143^2 = 48^2 + \cdots + 38^2 = 39^2 + \cdots + 7^2$.

oeis.org/A298467 is, smallest positive integer that is a sum of consecutive squares in exactly k ways. There are entries up to $k = 4$, so probably no one has found an integer that is a sum of consecutive squares in 5 ways. (Do any exist? If so, they exceed 10^{15}.) The $k = 4$ entry is $554503705 = 5 \times 7 \times 17 \times 43 \times 21673$.

oeis.org/A296338 is number of representations of n as a sum of consecutive squares.

024:18 (Gerhard Paseman) Let $L(n)$ be the (lesser known) Jacobsthal function: For $n > 1$, $L(n)$ is the length of the longest interval of consecutive integers sharing a common factor with n. It is enough to have $n \in SF$, the set of squarefree numbers > 1.

* Short version: where are these intervals of length $L(n)$? Where is the one closest to zero? By symmetry of $(j, n-j)$, for large odd n there is one in $(0, n/2)$. (Note: $n = 2p$ spoils this by having its witness in the middle.) How much better can we do?

* Long version: Let SFT (for Squarefree and Thin) be $n \in SF$ such that $\omega(n) = k < p(n)$ = least prime factor of n. We will look at $SFTO$, where k is actually odd. We have $L(n) = k$ for $n \in SFT$.

* Example: $385 = 5 \cdot 7 \cdot 11$ is in $SFTO$. $L(n) = 3$. By trial and error, we find $(20, 21, 22)$, and also $(98, 99, 100)$. We also have $(21 + 98, 20 + 100, 22 + 99) = (119, 120, 121)$ is a witness to L(n)=3. By symmetry, there are three other witnesses in $(0, 385)$, and similar and more addition relations exist.

* This example is isomorphic to *additive permutations:* write a permutation of the $k = 2j + 1$ symbols $(-j, \dots, 0, \dots, j)$ as a vector and do coordinate-wise addition. The above can be represented as $(-1, 0, 1) + (0, 1, -1) = (-1, 1, 0)$, which is a permutation of the same numbers. (One has to translate and then unscramble the output: $x - 1$ is a multiple of 5, $x + 0$ is a multiple of 7, and $x + 1$ is a multiple of 11.) This structure allows us a better upper bound (for $k = 3$ and $n \in SFTO$) of least witness coming before $n/3$ (improving on $n/2$).

* Larger examples exist for $n \in SFTO$ and odd k, with the structure giving a better upper bound on the location of the smallest witness. Subquestion 1: compute this improved bound for odd k.

* THIS BREAKS FOR EVEN k! No additive permutations. For $k = 4$ and 6, there are n where the least witness is close to $n/2$ and not to 0.

* Subquestion 2: find an inclusive notion that handles even k and generalizes additive permutations (or replaces them with something that looks like additive permutations for odd k and $n \in SFT$). Can we then use this to study the problem for $n \in SF$?